On lifetime optimization of Boolean systems with Erlang repair distributions

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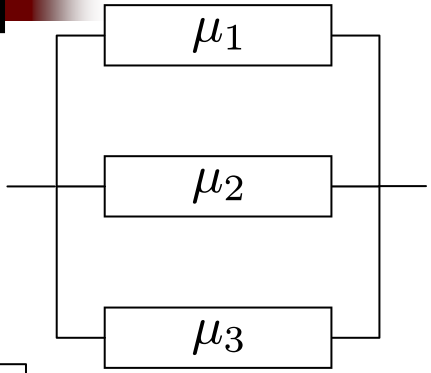
Outline

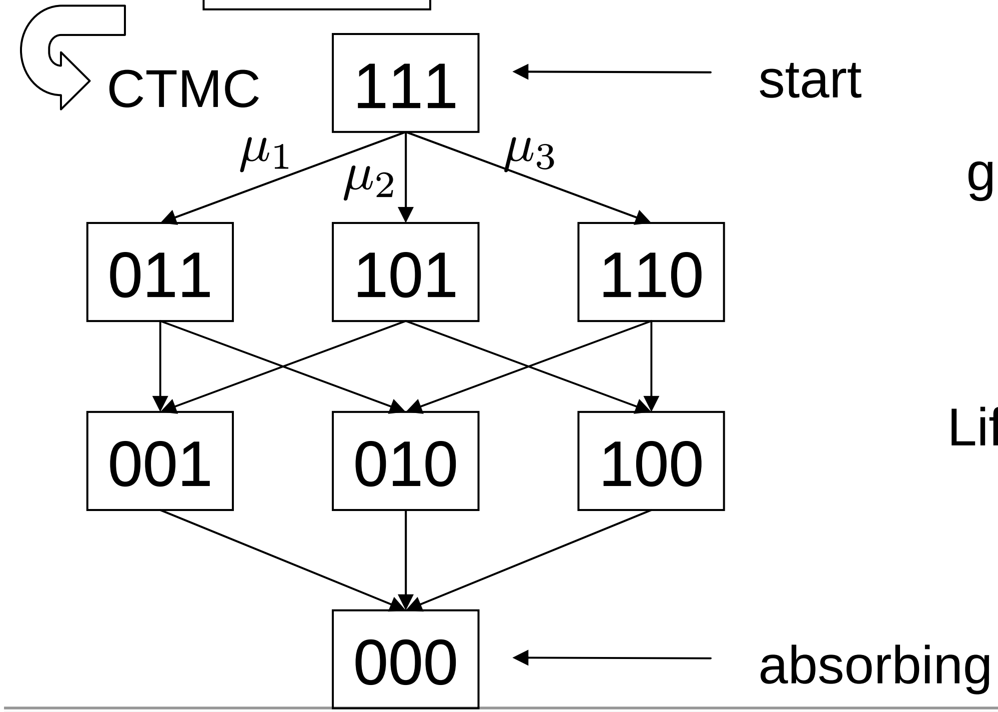
- Introduction
- Exponential repair distributions
- Erlang repair distributions with different semantics
    - Semi-Markov Model
    - Generalized Semi-Markov Model
- Toy Example

Boolean systems



Parallel system of N independent components

Component i has exp. distr. life time with rate μ_i



generator matrix

$$Q = \begin{pmatrix} S & s \\ 0 & 0 \end{pmatrix}$$

Lifetime of system is PH-distributed

$$\mathbb{P}(T_{Sys} \leq t) = 1 - \alpha e^{St}\mathbf{1}$$

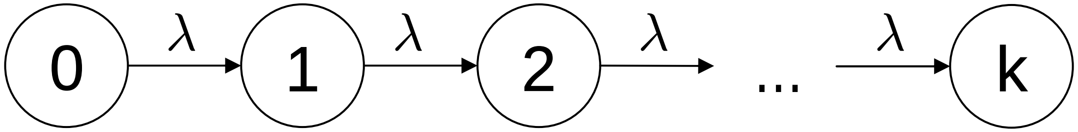
$$MTTF = \mathbb{E}(T_{Sys}) = -\alpha S^{-1}\mathbf{1}$$

Repair distributions

Goal: Maximize MTTF

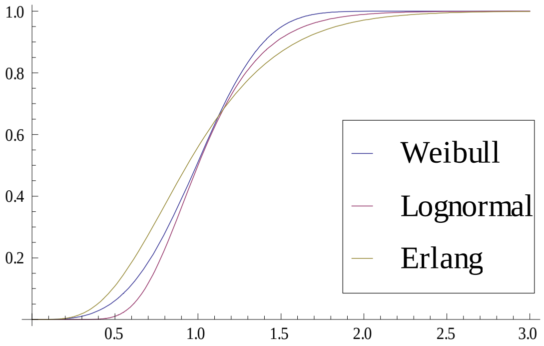
- Scheduler can assign a repairman to failed components

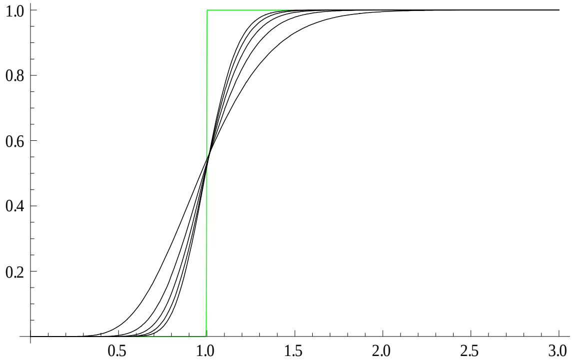
- Assume repairman with Erlang-k distributed repair time (time from state 0 to state k)



→ Approximation of typical repair time distributions, like

  - Weibull
  - Lognormal
  - Deterministic

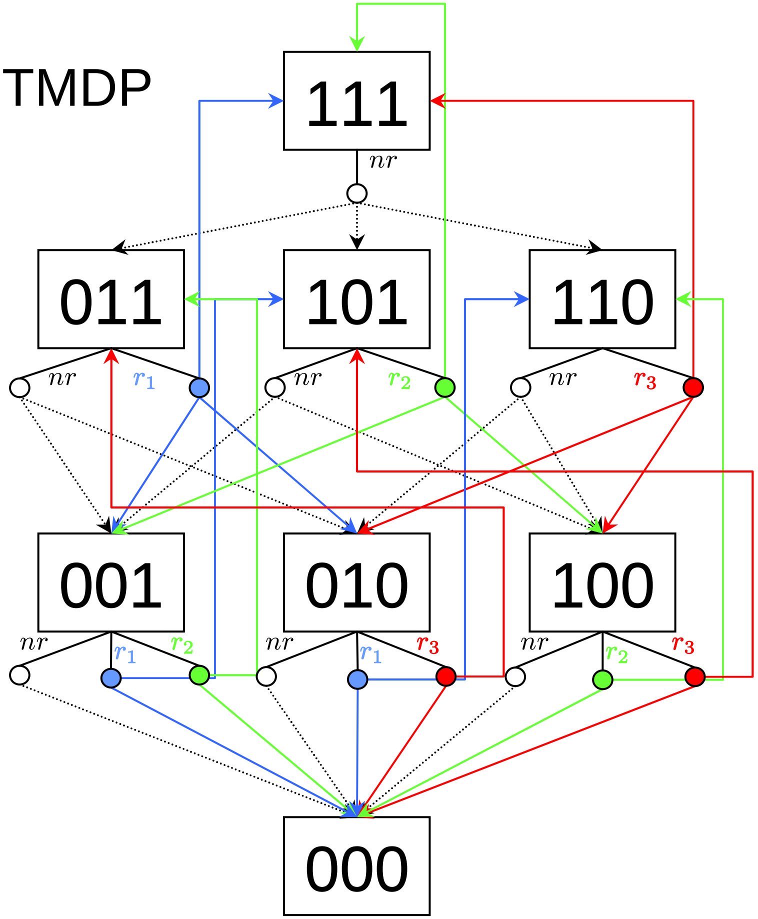




- (Extended) Preemptive repair: If during repair a further component fails the scheduler can reassign the repairman to any failed component

Special case: Exponential repair

CTMDP



State space: $S = \{0,1\}^N$

$$C_0(x) := \{i \mid x_i = 0\} \quad C_1(x) := \{i \mid x_i = 1\}$$

Action sets:

$$Act(x) = \{r_i \mid i \in C_0(x)\} \cup \{nr\}$$

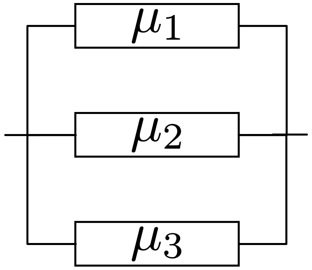
Rewards:

$$R(x,a) = \frac{1}{E(x,a)}$$

mean sojourn time

Solution to Exponential repair

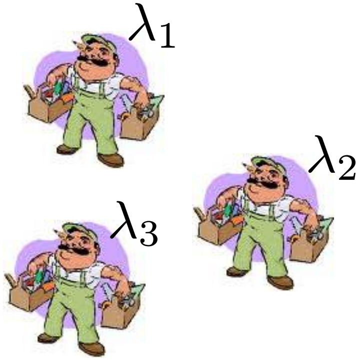
Katehakis and Melolidakis (1988)



N components
(exp. life time)

⟺

R repairmen
(exp. repair time)



+ preemption

Optimal scheduling policy:

Fastest **R**epairman – **M**ost **R**eliable **C**omponent

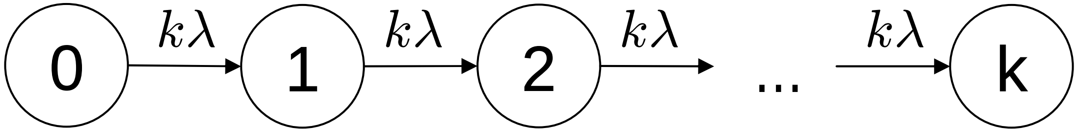
for the following objectives

- maximize the number of working components
- in a K-out-of-N system: maximize availability
- in a K-out-of-N system: maximize MTTF

Erlang repair semantics

Generalize exponential repair distributions to
Erlang repair distributions with same expected value

$$\mathbb{E}(T_{E_k}) = \mathbb{E}(T_{E_1}) = \frac{1}{\lambda}$$



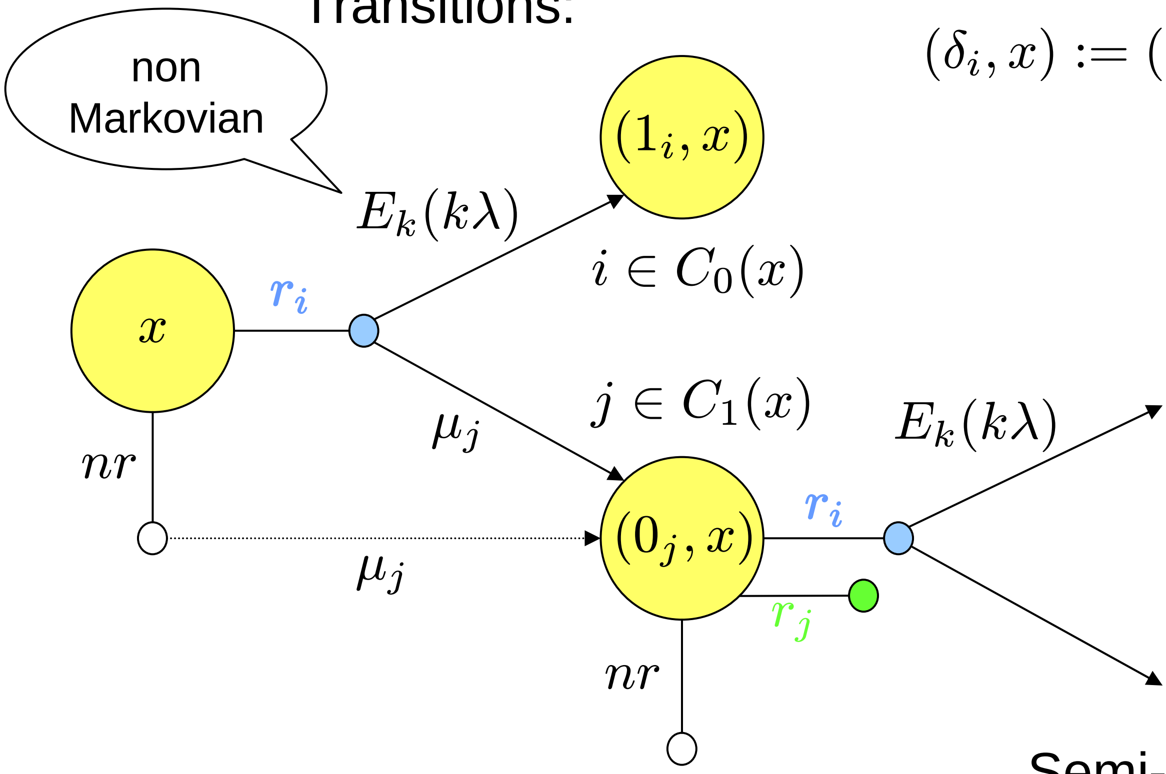
Semi-Markov Decision Model

State space: $S = \{0,1\}^N$

Action sets: $Act(x) = \{r_i \mid i \in C_0(x)\} \cup \{nr\}$

Transitions:

$$(\delta_i, x) := (x_1, \ldots, x_{i-1}, \delta, x_i, \ldots, x_N)$$



Transformation to

CTMDP

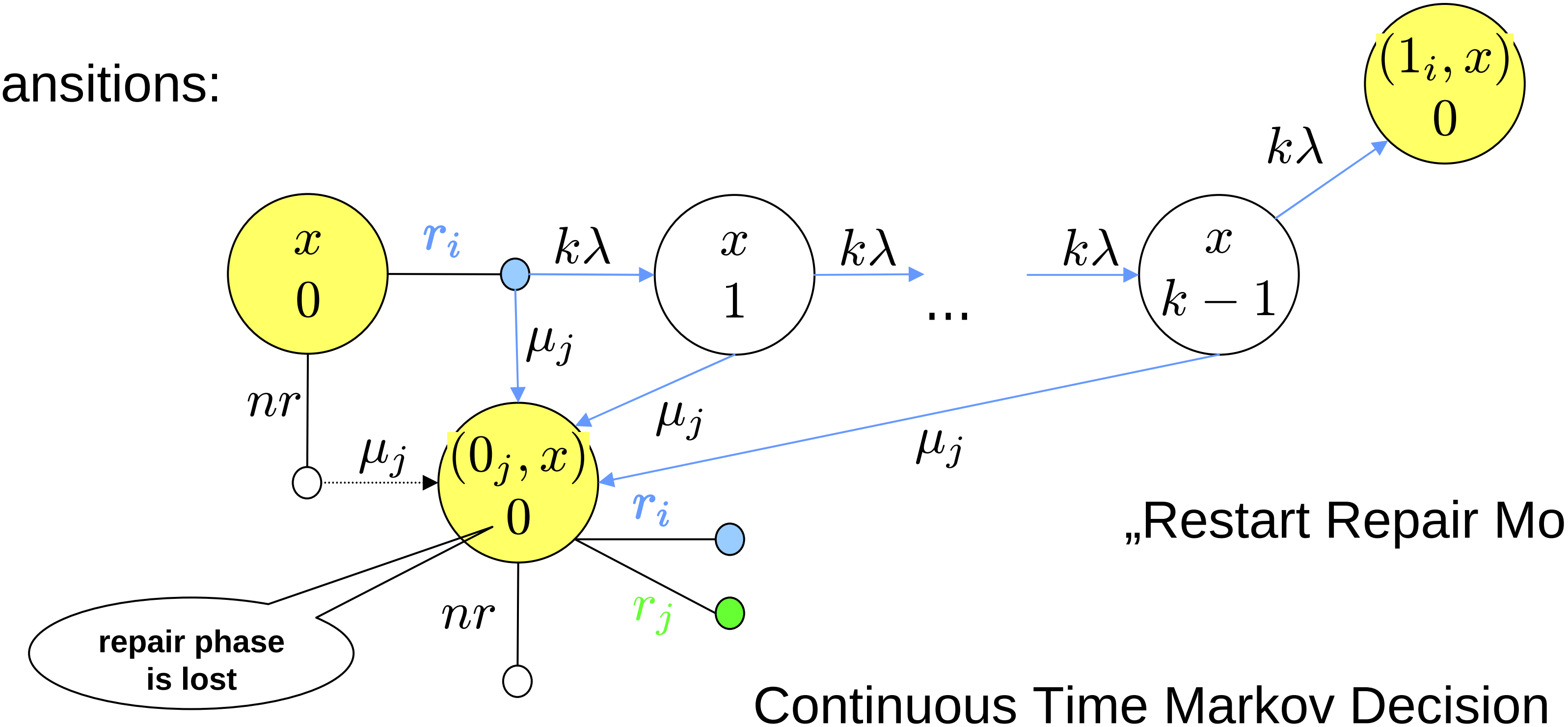
Semi-Markov Decision Process

Semi-Markov Decision Model

State space: $S_1 = \{(x,e) \in S \times \{0,1,\ldots,k-1\}^N \mid x_i = 1 \ \Rightarrow e_i = 0,\ \exists^{\leq 1} e_i \neq 0\}$

Action sets: $Act(x,e) = \begin{cases} Act(x), & e = (0,\ldots,0) \\ \{r_i\}, & e = (0,\ldots,e_i \neq 0,\ldots,0) \end{cases}$

Transitions:

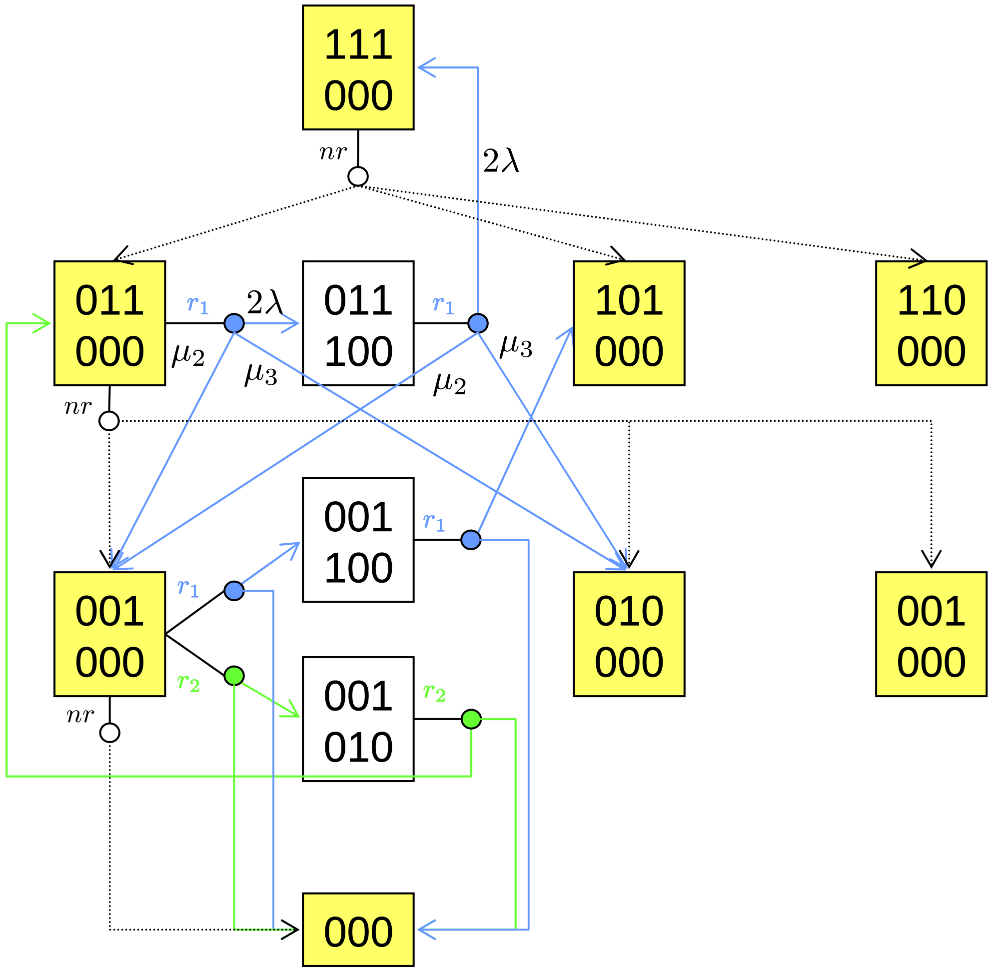


„Restart Repair Model“

Continuous Time Markov Decision Process

Restart Repair Model

Example: $N = 3$ $k = 2$



0 failed

1 failed

2 failed

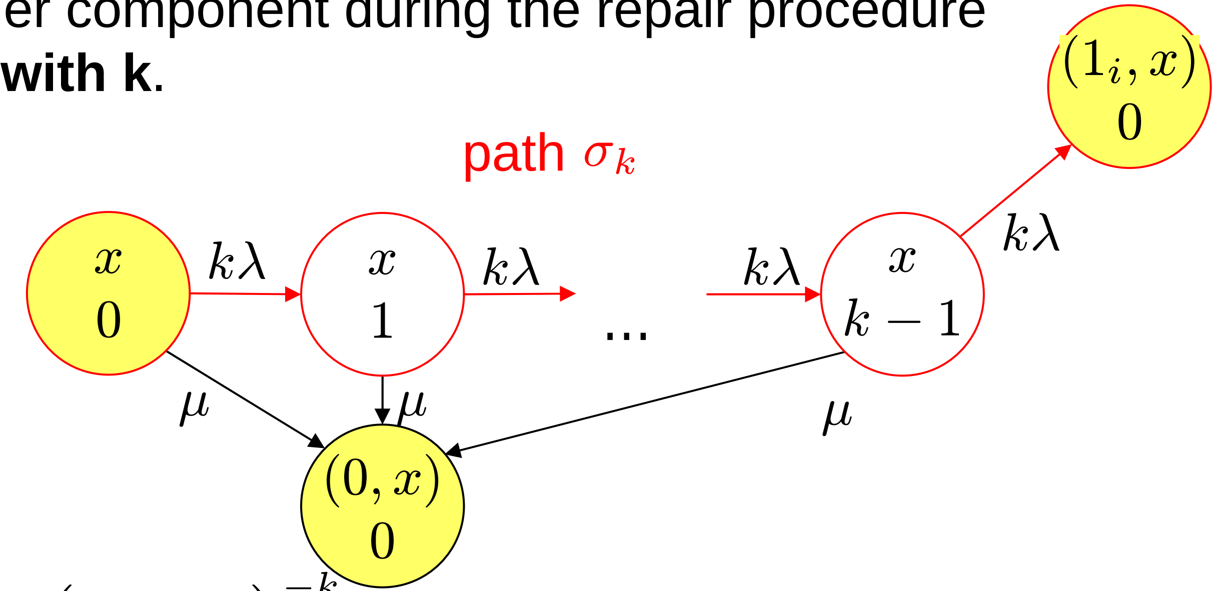
all failed

Restart Repair Model (SMDP)

Consider a sequence of Erlang-k distributed repair time T_{E_k} with

$$\mathbb{E}(T_{E_k}) = \frac{1}{\lambda}\ \forall k \in \mathbb{N}.$$

(1) The **probability** P_k to repair a failed component without further failure of another component during the repair procedure is **decreasing with k**.

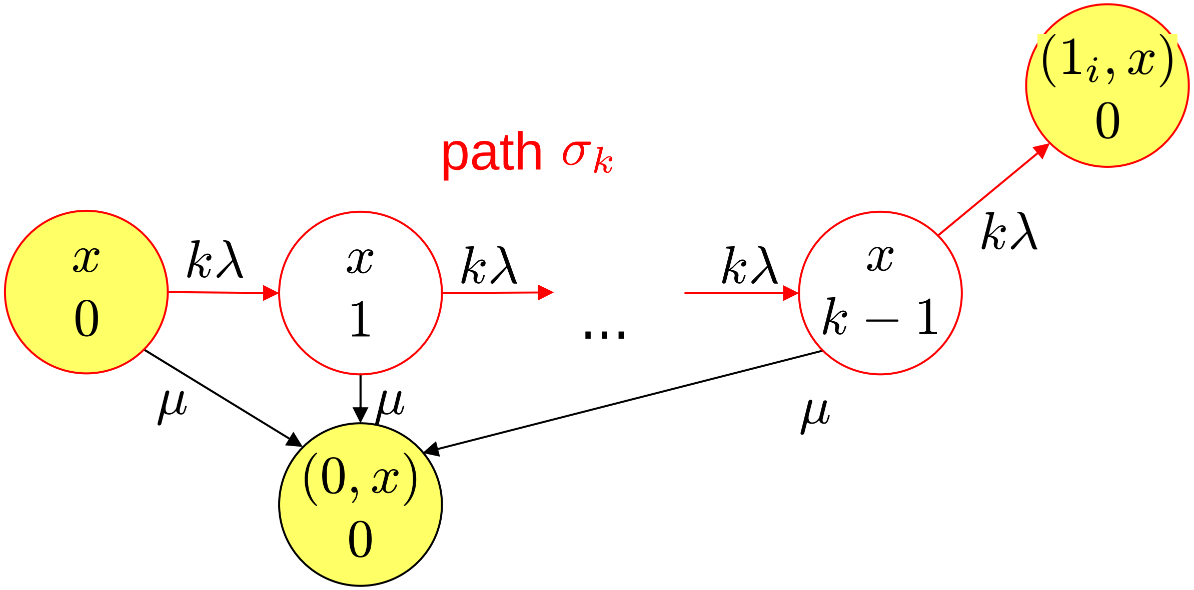


$$\mu = \sum_{j \in C_1(x)} \mu_j$$

$$P_k = \left(\frac{k\lambda}{k\lambda+\mu}\right)^k = \left(1 + \frac{\mu/\lambda}{k}\right)^{-k} \to e^{-\mu/\lambda} \text{ (decreasing)}$$

Restart Repair Model (SMDP)

(2) The **mean time** R_k to repair a failed component, under the condition that no further failure of another component during repair procedure occurs, is **increasing with k**.



$$R_k = \mathbb{E}(\sigma_k) = k \cdot \frac{1}{k\lambda+\mu} \to \frac{1}{\lambda} \text{ (increasing)}$$

Restart Repair Model (SMDP)

Special cases:

Among all Erlang distributions with equal expected values
(1) an exponentially distributed $(k = 1)$ repair time

- maximizes the probability to repair a failed component and
- minimizes the mean time to repair it

(2) deterministic repair time $(k \to \infty)$

- minimizes the probability to repair a failed component and
- maximizes the mean time to repair it

before a further component fails.

Does restart repair model hold in practice?

Erlang phases shall be used to model the repair progess!

Generalized Semi-Markov Process

Matthes (1962)

Generalized Semi-Markov Process (GSMP) =
Model for asynchronous events with arbitrary delay distributions

- state space S
- set of asynchronous events E (random delay)
- $E_s \subset E$: enabled events in state s
- transition probabilities for each event e

Semantics:

- enabled events in a state race to trigger first
- if a non-triggered event remains enabled in the next state, the original delay gets shortened by the time already spent (simulative approach)

Generalized SMDP

Younes and Simmons (2004): Extension to non-determinism

Generalized Semi-Markov Decision Process (GSMDP)

→ Split set of events into
- controllable events (actions) and
- exogenous events

GSMDP → exp. GSMDP → CTMDP

Assumptions:
- Factored state space S (defined by state variables)
- PH-distributed delays

Resolution of PH-delays by adding for each (non-exp.) event
- new state variables (representing phases)
- new exp. events (for each transition between phases)

Generalized SMDP

Example: N = 3

System state variables: $V = \{x_1, x_2, x_3\}$

effect

System events (failure): $\mathrm{f}_i = \langle x_i = 1, Exp(\mu_i), x_i \leftarrow 0 \rangle \quad i = 1, 2, 3$

enabling condition

delay distribution

Repair events: $\mathrm{r}_i = \langle x_i = 0, E_k(k\lambda), x_i \leftarrow 1 \rangle \quad i = 1, 2, 3$

Transformation to exp. GSMDP ⇩

phase of r_i

New state variables: $\hat{V} = V \cup \{e_1, e_2, e_3\}$

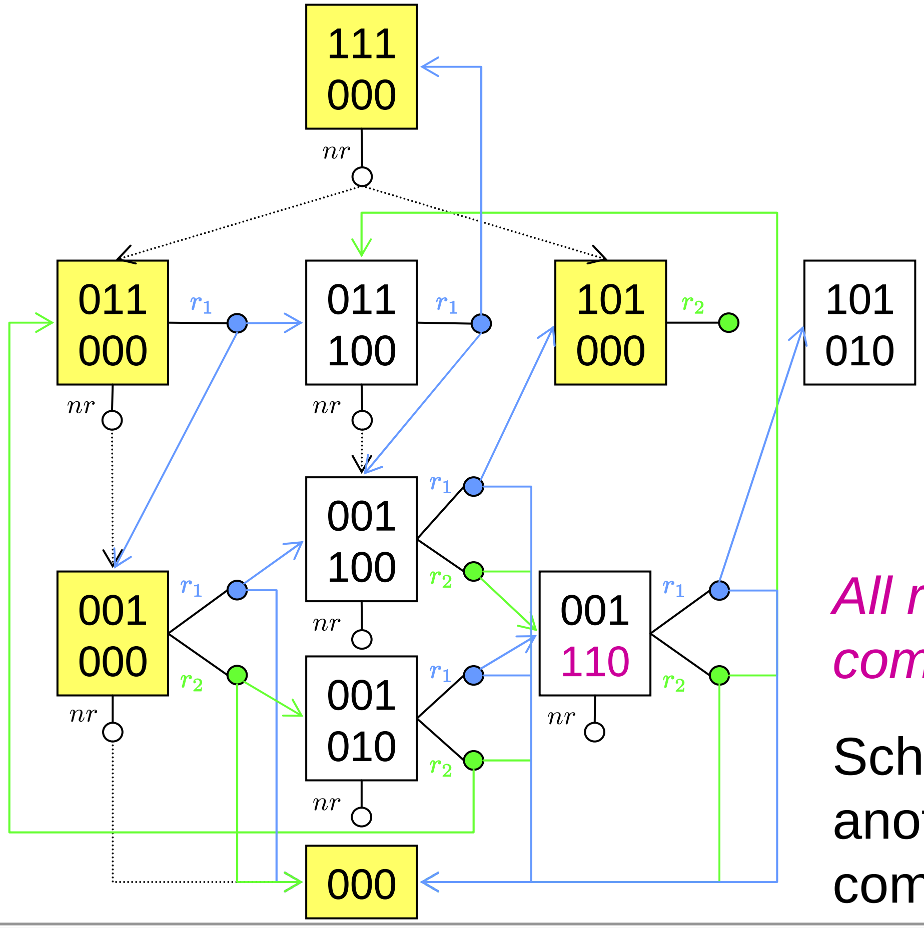
Repair events: $\mathrm{r}_i^{(j)} = \langle x_i = 0 \wedge e_i = j - 1, Exp(k\lambda), e_i \leftarrow j \rangle, \quad j \leq k - 1$

$$\mathrm{r}_i^{(k)} = \langle x_i = 0 \wedge e_i = k - 1, Exp(k\lambda), x_i \leftarrow 1 \wedge e_i \leftarrow 0 \rangle$$

The non-repair action nr transforms to an idle action

Generalized SMDP

exp. GSMDP ⟹ CTMDP



„Full Memory Model“

All repair phases of all partially repaired components are remembered.

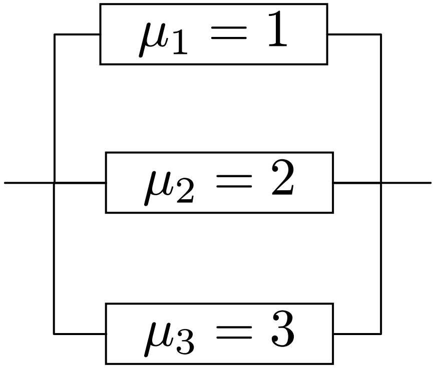
Scheduler can reassign the repairman to another component, even if no further component fails during repair procedure.

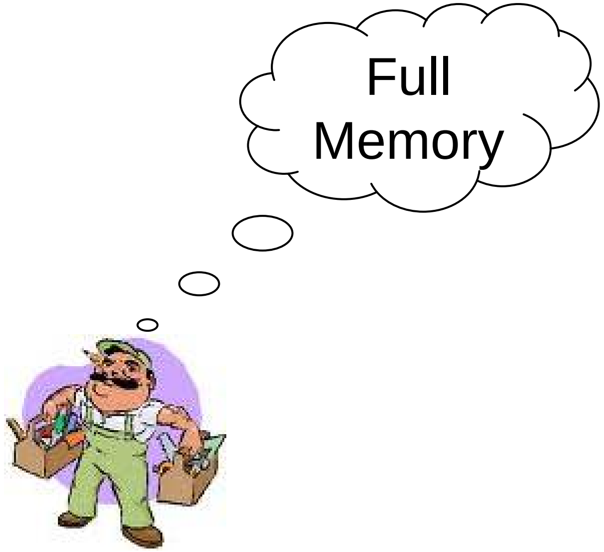
Full Memory Model

Is there an optimal rule, like the FR-MRC policy, for the case of Erlang-k repair distributions with full memory semantics?

Counterexample:

3 components

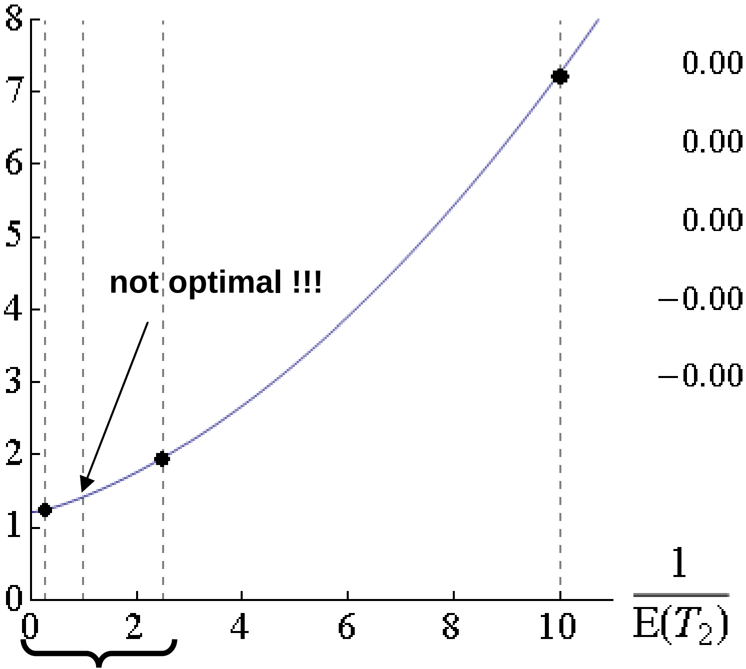




Erlang-2 repairman

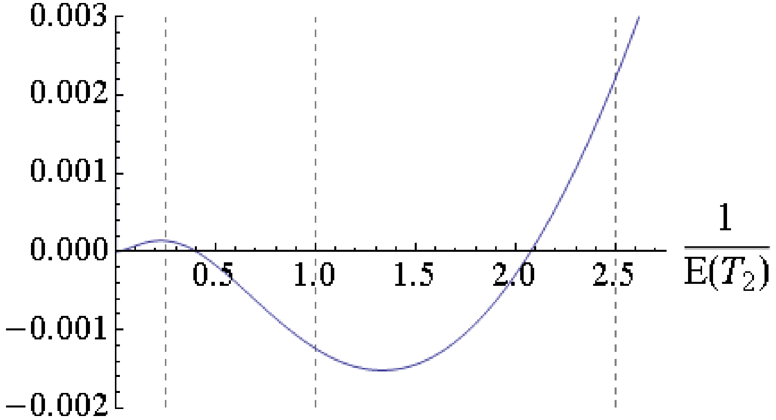
Counterexample

mean repair time	4	1	0.4	0.1
MTTF for policy I	1.2444	1.4279	1.9663	7.2607
MTTF for policy II	1.2442	1.4292	1.9640	7.0874
optimal policy	I	II	I	I



not optimal !!!

MTTF for policy I



Difference of MTTF values for policies I and II

state \ policy	I	II
011-000	1	1
011-100	1	1
101-000	2	2
101-010	2	2
110-000	3	3
110-001	3	3
001-000	1	1
001-100	1	1
001-010	2	2
001-110	▨	▨
010-000	1	1
010-100	1	1
010-001	**3**	**1**
010-101	▨	1
100-000	2	2
100-010	2	2
100-001	3	3
100-011	▨	▨

not reachable

Future Work

(1) dynamic programming methods need to work iteratively
with State→Value function

How can one save this function efficiently to memory?
→ Approximation techniques (least squares, aggregation)

Tune dynamic programming iterations
→ Hybrid optimization: partial simulation + analytic methods

(2) starting heuristics: importance measures (Barlow-Proschan, Natvig)

Thank you for your attention!

References

(1) Smith – Optimal repair allocation of a series system, *Operations Research*, 26(4):653-662, 1978

(2) Derman, Katehakis – Optimal repair allocation in a series system *Mathematics of Operations Research*, 9(4):615-623, 1984

(3) Katehakis, Melolidakis – Dynamic repair allocation for a k-out-of-n system maintained by distinguishable repairmen, *Probability in the Engineering and Informational Sciences*, 2:51-62, 1988

(4) Puterman – Markov Decision Processes, *John Wiley & Sons INC*, 1994

(5) Azadmanesh, Hui, Kieckhafer – On the sensitivity of NMR unreliability to non-exponential repair distributions, *Fifth International Symposium on High Assurance Systems Engineering*, 293-300, 2000

(6) Simmons, Younes – Solving Generalized Semi-Markov Processes using Continuous Phase-Type Distributions, *Proceedings of the Nineteenth National Conference on Artificial Intelligence*, 742-747, 2004