Discrete-Time and Continuous-Time Markov Chains

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Probabilistic Model Checking for Discrete-Time Markovian Models

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Probability elsewhere

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- randomized algorithms [Rabin 1960]
  symmetry breaking, finderprint techniques,
  random choice of waiting times or IP addresses, ...
- stochastic control theory [Bellman 1957]
  operations research
- performance modeling [Markov, Erlang, Kolm., ∼ 1900]
- biological systems
- resilient systems

  ⋮

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discrete or continuous-time Markovian models

memoryless property: future system behavior depends only on the current state, but not on the past

Probabilistic models

Probabilistic models

	purely probabilistic	probabilistic and nondeterministic
discrete time		
continuous time		

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discrete time	discrete-time Markov chain (DTMC)	Markov decision process (MDP)
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Probabilistic models

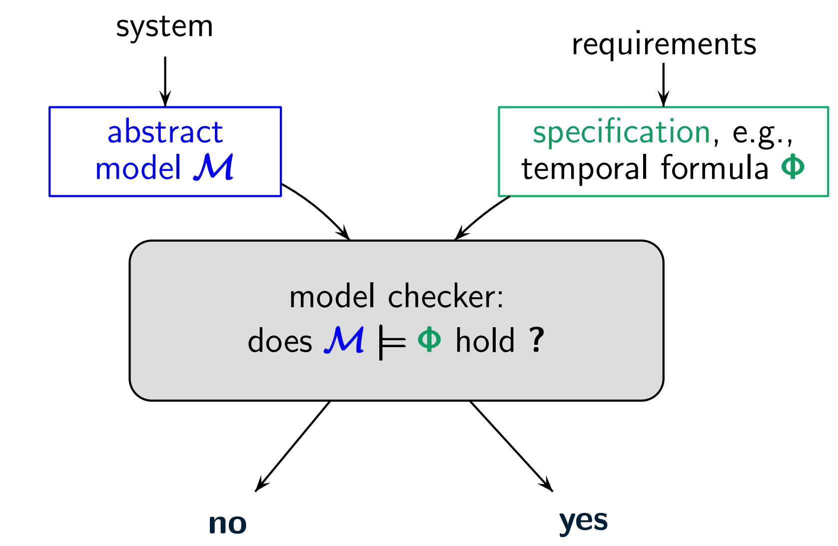
	purely probabilistic	probabilistic and nondeterministic
discrete time	discrete-time Markov chain (DTMC)	Markov decision process (MDP)
continuous time	continuous-time Markov chain (CTMC)	various models

↑ tutorial by *Marta Kwiatkowska*

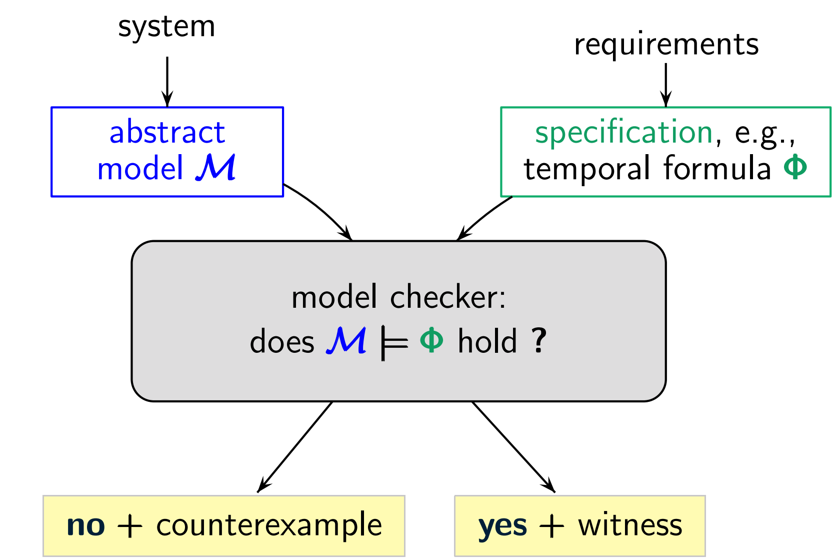
Probabilistic models

	purely probabilistic	probabilistic and nondeterministic
discrete time	discrete-time Markov chain (DTMC)	Markov decision process (MDP)
continuous time	continuous-time Markov chain (CTMC)	continuous-time MDP interactive Markov chains probabilistic timed automata stochastic automata ⋮

Model checking



Model checking



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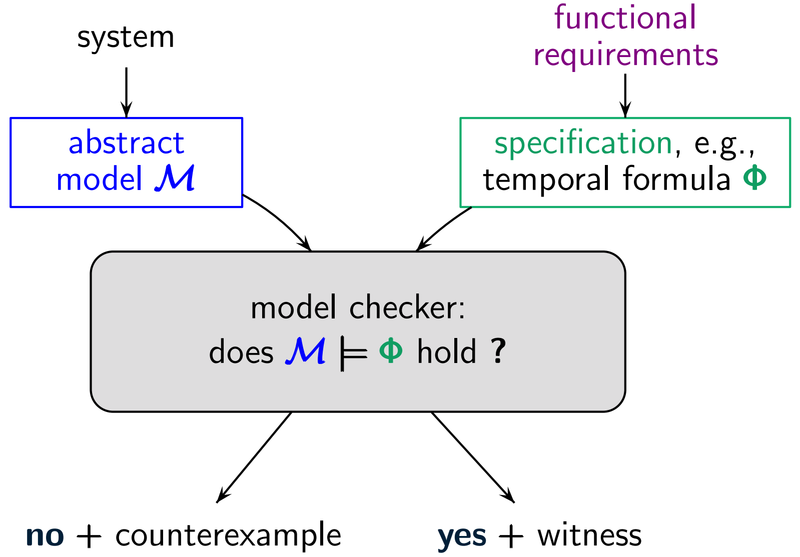
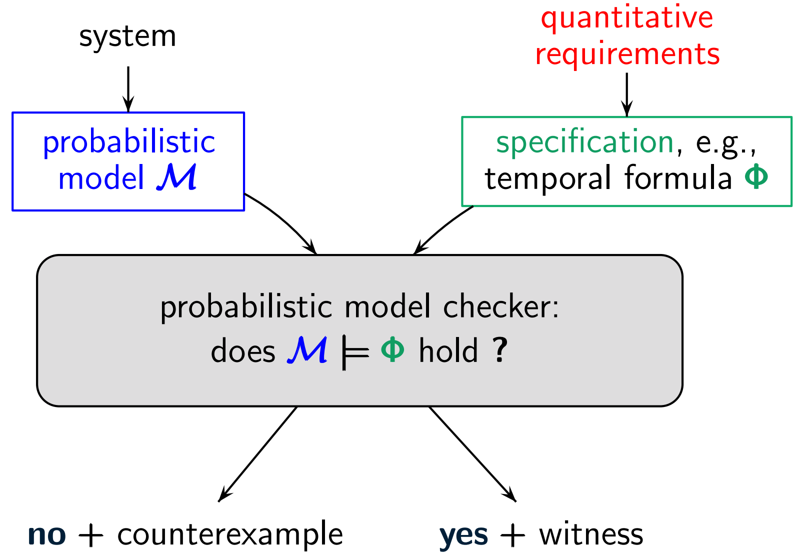


Probabilistic model checking



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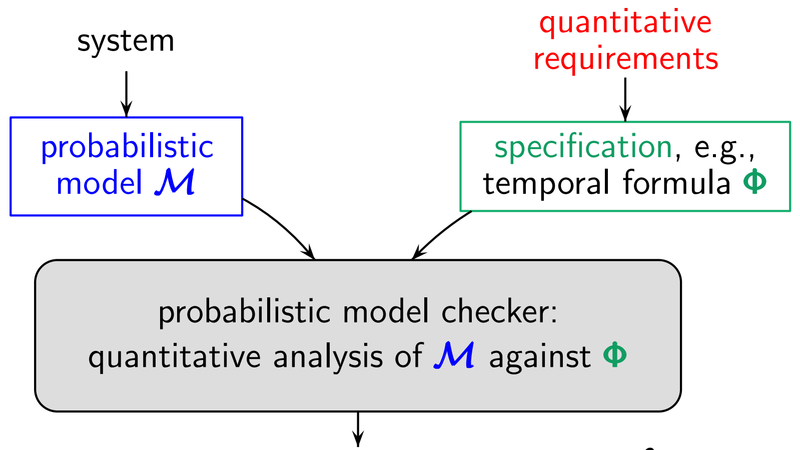
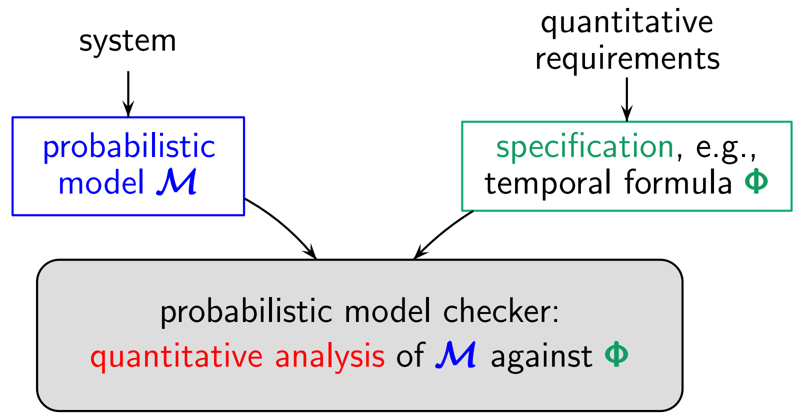


probability for "bad behaviors" is $< \mathbf{10^{-6}}$
probability for "good behaviors" is **1**
expected costs for ....

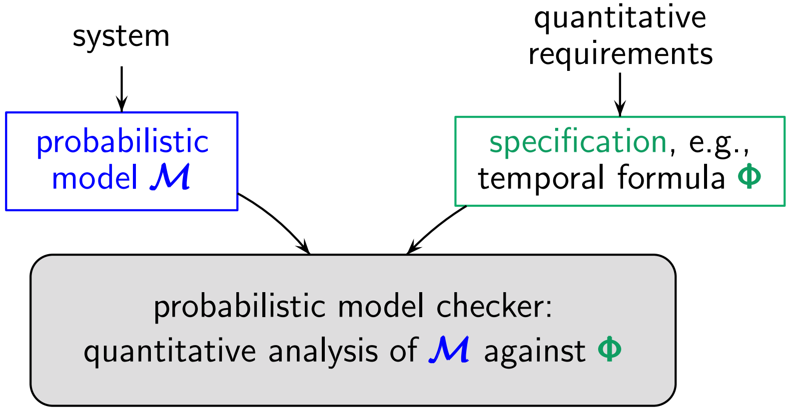
Probabilistic model checking



quantitative analysis relies on a combination of

- model checking techniques
- known concepts for probabilistic models

Probabilistic model checking



logical approach $\leadsto$ unambiguous specifications

model checking $\leadsto$ automatic reasoning

Discrete-time Markovian models

Part 1: discrete-time Markov chains (DTMC)

1. basic definitions
2. probabilistic computation tree logic (PCTL/PCTL*)
3. expected rewards

Part 2: Markov decision processes (MDP)

1. basic definitions
2. PCTL/PCTL* model checking
3. fairness

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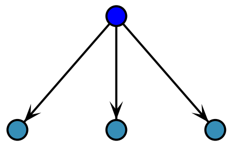
Markov chains

... transition systems with probabilistic distributions for the successor states

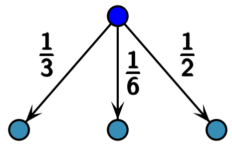
Markov chains

DTMC-05

... transition systems with probabilistic distributions for the successor states



transition system: nondeterministic branching



Markov chain: probabilistic branching

(discrete-time)

Discrete-time Markov chain (DTMC)

DTMC-07

$\mathcal{M} = (S, P, \ldots)$

Discrete-time Markov chain (DTMC)

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$\mathcal{M} = (S, P, \ldots)$

- state space S

Discrete-time Markov chain (DTMC)

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$\mathcal{M} = (S, P, \ldots)$

- state space S ⟵ here: finite

Discrete-time Markov chain (DTMC)

$\mathcal{M} = (S, P, \ldots)$

- state space S ⟵ here: finite
- transition probability function $P : S \times S \to [0,1]$

  s.t. $\sum_{s' \in S} P(s, s') = 1$

Discrete-time Markov chain (DTMC)

$\mathcal{M} = (S, P, \ldots)$

- state space S ⟵ here: finite
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discrete-time or time-abstract:
probability for the step $s \longrightarrow s'$

Discrete-time Markov chain (DTMC)

$\mathcal{M} = (S, P, AP, L, \ldots)$

- state space S
- transition probability function $P : S \times S \to [0,1]$

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- AP set of atomic propositions
- labeling function $L : S \to 2^{AP}$

Discrete-time Markov chain (DTMC)

$\mathcal{M} = (S, P, AP, L, \ldots)$ ⟵

+ initial distribution
+ reward/cost function
+ ...

- state space S
- transition probability function $P : S \times S \to [0,1]$

  s.t. $\sum_{s' \in S} P(s, s') = 1$
- AP set of atomic propositions
- labeling function $L : S \to 2^{AP}$
- $\mu : S \to [0,1]$ initial distribution
- $rew : S \to \mathbb{N}$ where $rew(s)$ is the reward earned per visit of state s

Example: DTMC for communication protocol

$\mathcal{M} = (S, P, AP, L, \ldots)$

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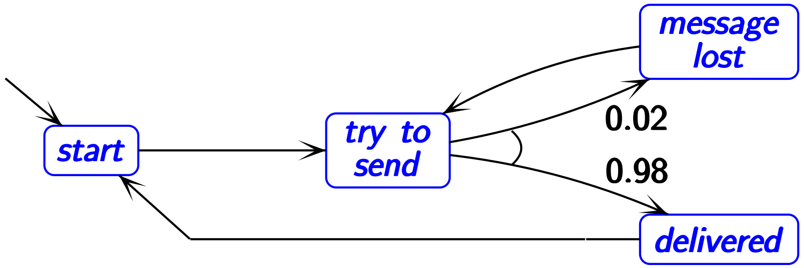

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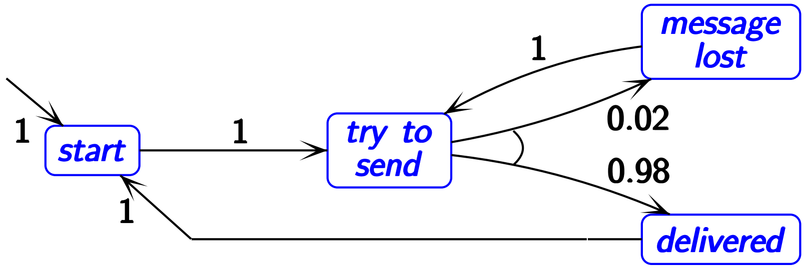

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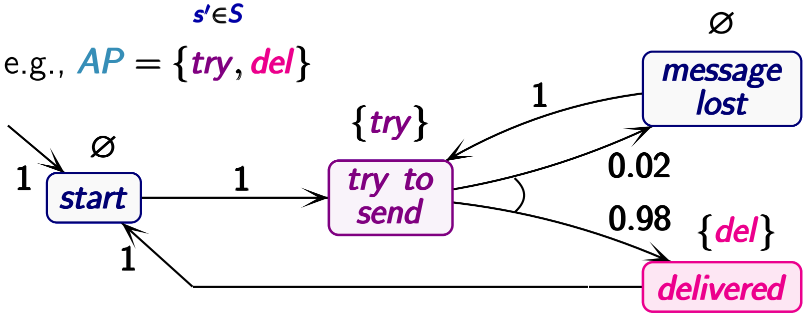
e.g., $AP = \{try, del\}$



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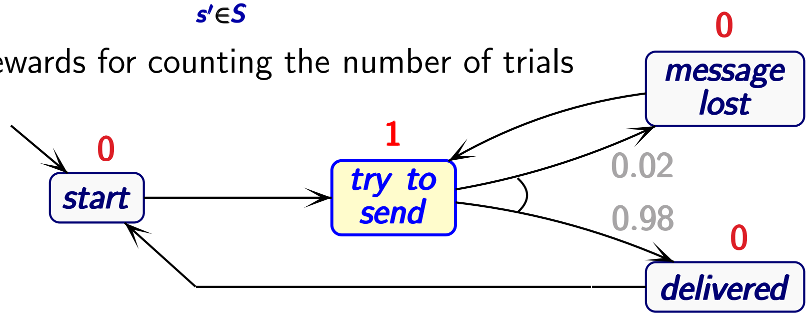
rewards for counting the number of trials



Probability measure of a Markov chain

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$\mathcal{M} = (S, P, AP, L, \mu)$ Markov chain

↑ initial distribution $\mu : S \to [0,1]$

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probability measure for (measurable) sets of paths:

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probability measure for (measurable) sets of paths:

consider the σ-algebra generated by cylinder sets

$\Delta(s_0\, s_1 \ldots s_n)$ = set of infinite paths $s_0\, s_1 \ldots s_n\, s_{n+1}\, s_{n+2}\, s_{n+3} \ldots$

↑ finite path

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$\Delta(s_0\, s_1 \ldots s_n) \;=\;$ set of infinite paths ...

σ-algebra on universe $\mathcal{U}$: set $\mathcal{V} \subseteq 2^{\mathcal{U}}$ s.t.

1. $\mathcal{U} \in \mathcal{V}$
2. if $T \in \mathcal{V}$ then $\mathcal{U} \setminus T \in \mathcal{V}$
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here: $\mathcal{U} \;=\;$ set of infinite paths $\subseteq S^\omega$

$\mathcal{V} \;=\;$ smallest subset of $2^{\mathcal{U}}$ that contains all cylinder sets and is closed under complement and countable unions

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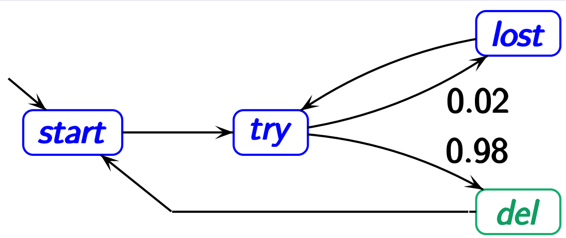
probability measure is given by:

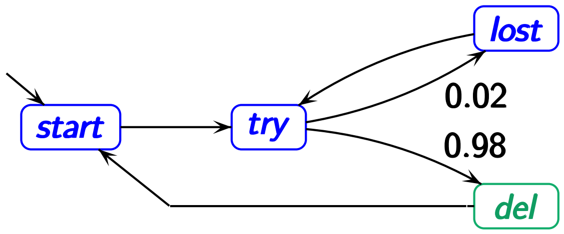
$$\Pr^{\mathcal{M}}\big(\,\Delta(s_0\, s_1 \ldots s_n)\,\big) \;=\; \mu(s_0) \cdot \prod_{1 \leqslant i \leqslant n} P(s_{i-1}, s_i)$$

Example: Markov chain



probability for delivering the message within **5** steps:

Example: Markov chain

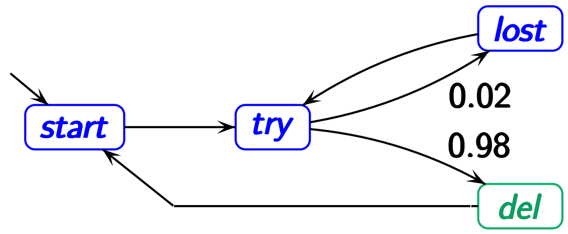


probability for delivering the message within **5** steps:

$$= \Pr^{\mathcal{M}}(\textit{start try del}) + \Pr^{\mathcal{M}}(\textit{start try lost try del})$$

notation: $\Pr^{\mathcal{M}}(s_0\, s_1 \ldots s_n) = \Pr^{\mathcal{M}}(\Delta(s_0\, s_1 \ldots s_n))$

Example: Markov chain



probability for delivering the message within **5** steps:

$$= \Pr^{\mathcal{M}}(\textit{start try del}) + \Pr^{\mathcal{M}}(\textit{start try lost try del})$$

$$= 0.98 + 0.02 \cdot 0.98 = 0.9996$$

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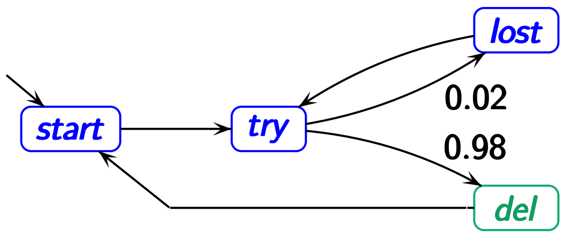
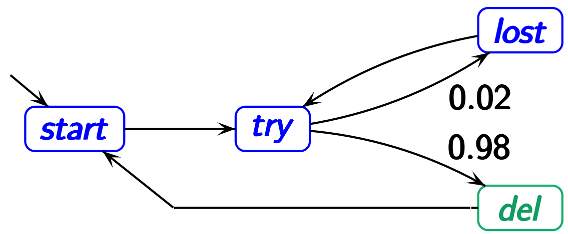
Example: Markov chain



probability for eventually delivering the message:

Example: Markov chain

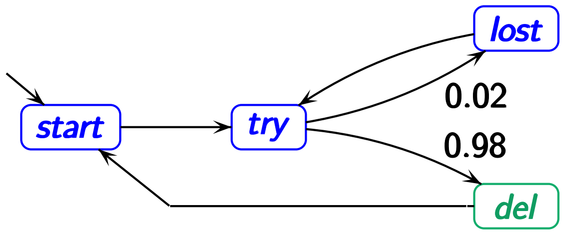


probability for eventually delivering the message:

$$= \sum_{n=0}^{\infty} \Pr^{\mathcal{M}}(\textit{start try } (\textit{lost try})^n \textit{ del})$$

Example: Markov chain

DTMC-20



probability for eventually delivering the message:

$$= \sum_{n=0}^{\infty} \Pr^{\mathcal{M}}(\mathit{start}\ \mathit{try}\ (\mathit{lost}\ \mathit{try})^n\ \mathit{del})$$

$$= \sum_{n=0}^{\infty} 0.02^n \cdot 0.98 \ = \ 1$$

Measurability of classical properties

DTMC-25

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A σ-algebra is a pair $(\mathcal{U}, \mathcal{V})$ where $\mathcal{U}$ is a set and $\mathcal{V} \subseteq 2^{\mathcal{U}}$ s.t.

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The elements of $\mathcal{V}$ are called measurable events.

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For DTMCs: $\mathcal{U}$ = set of infinite paths

$\mathcal{V}$ = σ-algebra generated by the cylinder sets

$$\Delta(s_0\, s_1 \ldots s_n) = \left\{ \begin{array}{l} \text{set of infinite paths } \pi \text{ of the form} \\ s_0\, s_1 \ldots s_n\, s_{n+1}\, s_{n+2}\, s_{n+3} \ldots \end{array} \right.$$

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step-bounded reachability: "visit G within n steps"

$$\Diamond^{\leqslant n} G = \bigcup_{0\leqslant i\leqslant n} \bigcup_{s_0,\ldots,s_i} \Delta(s_0\, s_1 \ldots s_{i-1}\, s_i)$$

where $s_i \in G$

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$$\Pr^{\mathcal{M}}(\Diamond^{\leqslant n} G) = \sum_{0\leqslant i\leqslant n} \sum_{s_0,\ldots,s_i} \Pr^{\mathcal{M}}(s_0\, s_1 \ldots s_{i-1}\, s_i)$$

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unbounded reachability: "visit G eventually"

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$$\Pr^{\mathcal{M}}(\Diamond G) \;=\; \sum_{i \in \mathbb{N}} \; \sum_{s_0,\ldots,s_i} \Pr^{\mathcal{M}}(s_0\, s_1 \ldots s_{i-1}\, s_i)$$

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repeated reachability: "visit G infinitely often"

$$\Box\Diamond G \;=\; \bigcap_{n \in \mathbb{N}} \; \bigcup_{i \geqslant n} \; \bigcup_{s_0,\ldots,s_i} \Delta(s_0\, s_1 \ldots s_{i-1}\, s_i)$$

where $s_i \in G$

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where $s_i \in G$, but possibly $s_j \in G$ for some $j < i$

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persistence: "from some moment on always G"

$$\Diamond\Box G \;=\; Paths^{\mathcal{M}} \setminus \Box\Diamond\neg G$$

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persistence: "from some moment on always G"

$$\Diamond\Box G \;=\; Paths^{\mathcal{M}} \setminus \Box\Diamond\neg G$$

$$\Pr^{\mathcal{M}}(\Diamond\Box G) \;=\; 1 - \Pr^{\mathcal{M}}(\Box\Diamond\neg G)$$

Stochastic process of a Markov chain

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stochastic process (general definition):

family $(X_t)_{t \in Time}$ of random variables $X_t : \mathcal{U} \to S$

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family $(X_t)_{t \in Time}$ of random variables $X_t : \mathcal{U} \to S$

- $Time$ is a time domain, e.g., $\mathbb{N}$ or $\mathbb{R}_{\geqslant 0}$
- S is a set (with fixed σ-algebra)
- $\mathcal{U}$ is a sample space (with fixed σ-algebra and probability measure)

Stochastic process of a Markov chain

stochastic process of DTMC $\mathcal{M} = (S, P, \ldots)$

family $(X_t)_{t \in Time}$ of random variables $X_t : \mathcal{U} \to S$

- $Time$ is a time domain $\longleftarrow$ $Time = \mathbb{N}$
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Markov property:

$$\Pr^{\mathcal{M}}\big(X_t = s \,\big|\, X_{t-1} = u \big) \; =$$
$$\Pr^{\mathcal{M}}\big(X_t = s \,\big|\, X_{t-1} = u,\, X_{t-2} = s_{t-2},\, \ldots,\, X_0 = s_0 \big)$$

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$$\Pr^{\mathcal{M}}\big(X_1 = s \,\big|\, X_0 = u\big) \qquad \text{time-homogeneous}$$

Transient and long-run distribution

Transient and long-run distribution

transient: ... refers to a fixed time point t

long-run: ... when time tends to infinity

Transient distribution

Let $\mathcal{M} = (S, P, \mu, \ldots)$ be a DTMC, $t \in \mathbb{N}$ and $s \in S$.

transient state probability:

$$\begin{aligned} \mu_t(s) &= \Pr^{\mathcal{M}}\{s_0\, s_1\, s_2 \ldots \in Paths^{\mathcal{M}} : s_t = s\} \\ &= \Pr^{\mathcal{M}}(X_t = s) \end{aligned}$$

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↑ initial distribution (row vector)

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↑ t-th power of transition probability matrix $P^t = P^{t-1} \cdot P$

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↑ unit vector $(0 \ldots 0, 1, 0, \ldots 0)^T$ representing Dirac distribution for state s

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transient state distribution
for time point $t-1$

$\mu_0 = \mu$ initial distribution

$\mu_n = \mu_{n-1} \cdot P$ for $n \geqslant 1$

Long-run distributions

Let $\mathcal{M} = (S, P, \mu, \ldots)$ be a DTMC.

steady-state probability: $\widetilde{\mu}(s) = \lim\limits_{t \to \infty} \mu_t(s)$

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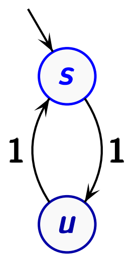

$\mu_{2t}(s) = 1$ ← even time points

$\mu_{2t+1}(s) = 0$ ← odd time points

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Let $\mathcal{M} = (S, P, \mu, \ldots)$ be a DTMC.

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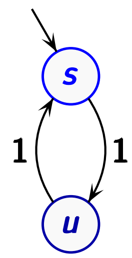
- limit may not exist or depend on the initial distribution μ



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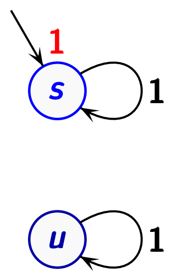

If $\mu(s) = 1$ then: $\tilde{\mu}(s) = 1$

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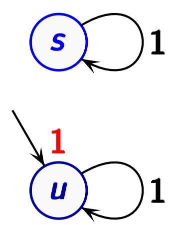

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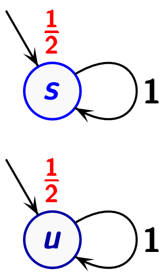

If $\mu(s) = 1$ then: $\widetilde{\mu}(s) = 1$

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If $\mu(s) = \mu(u) = \frac{1}{2}$ then:
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↑ balance equation

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↑ balance equation

unique solution with the side constraint

$$\sum_{s\in S} \widetilde{\mu}(s) = 1$$

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- Cesàro limit always exists
- if the steady-state probabilities exists: $\widetilde{\mu}(s) = \theta(s)$
- if $\mathcal{M}$ is strongly connected: θ is computable via the balance equation $\theta = P \cdot \theta$ where $\sum_{s\in S} \theta(s) \;=\; 1$

Fundamental property of finite Markov chains

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Almost surely, i.e., with probability **1**:

A bottom strongly connected component will be reached and all its states visited infinitely often.

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A bottom strongly connected component will be reached and all its states visited infinitely often.

$$Pr^{\mathcal{M}}\Big\{ s_0\, s_1\, s_2 \ldots \in Paths^{\mathcal{M}} : \text{there exists } i \geqslant 0 \text{ and a BSCC } C \text{ s.t. } \underbrace{\forall j \geqslant i.\ s_j \in C}_{\text{eventually forever } C} \ \wedge\ \underbrace{\forall s \in C\ \overset{\infty}{\exists} j.\ s_j = s}_{\text{visit each state in } C \text{ infinitely often}} \Big\} = 1$$

Fundamental property of finite Markov chains

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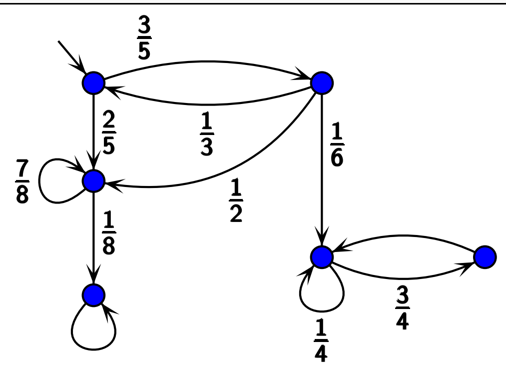

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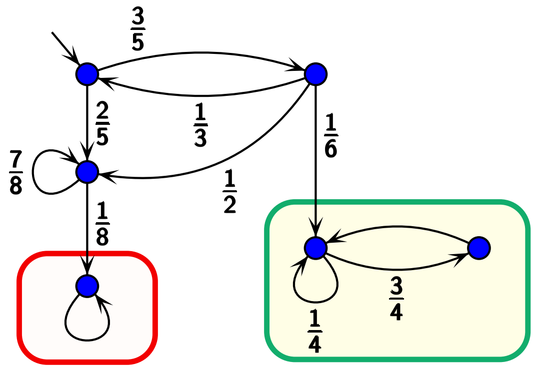

2 BSCCs

Fundamental property of finite Markov chains

Almost surely, i.e., with probability **1**:

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long-run distribution:

$\theta(s) > 0$ iff s belongs to some BSCC

Fundamental property of finite Markov chains

Almost surely, i.e., with probability **1**:

A bottom strongly connected component will be reached and all its states visited infinitely often.

long-run distribution:

If s is a state of BSCC B then:

$$\theta(s) = \Pr^{\mathcal{M}}(\Diamond B) \cdot \theta^{B}(s)$$

probability to reach B — long-run probability for s inside B

Discrete-time Markovian models

Part 1: discrete-time Markov chains (DTMC)

1. basic definitions
2. probabilistic computation tree logic (PCTL/PCTL*) ⟵
3. expected rewards

Part 2: Markov decision processes (MDP)

1. basic definitions
2. PCTL/PCTL* model checking
3. fairness

Probabilistic computation tree logic

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PCTL/PCTL* [Hansson/Jonsson 1994]

- probabilistic variants of CTL/CTL*
- contains a probabilistic operator $\mathbb{P}$
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Probabilistic computation tree logic

PCTL/PCTL* [Hansson/Jonsson 1994]

- probabilistic variants of CTL/CTL*
- contains a probabilistic operator $\mathbb{P}$
  to specify lower/upper probability bounds
- operators for expected costs, long-run averages, ...
  not considered here, but can be added

Syntax of PCTL*

state formulas:

$$\Phi ::= \mathit{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \quad \ldots$$

path formulas:

$$\varphi ::= \ldots$$

Syntax of PCTL*

state formulas:

$$\Phi \ ::= \ true \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \mathbb{P}_{\mathrm{I}}(\varphi)$$

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where $a \in AP$ is an atomic proposition

$\mathrm{I} \subseteq [0,1]$ is a probability interval

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PCTL-20

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$\mathrm{I} \subseteq [0,1]$ is a probability interval

qualitative properties: $\mathbb{P}_{>0}(\varphi)$ or $\mathbb{P}_{=1}(\varphi)$

quantitative properties: e.g., $\mathbb{P}_{>0.5}(\varphi)$ or $\mathbb{P}_{\leqslant 0.01}(\varphi)$

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↑ state formula

Syntax of PCTL* path formulas

PCTL-22

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path formulas:

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$\bigcirc \ \widehat{=}$ next

Syntax of PCTL* path formulas

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$\bigcirc \mathrel{\widehat{=}}$ next $\qquad \mathsf{U} \mathrel{\widehat{=}}$ until

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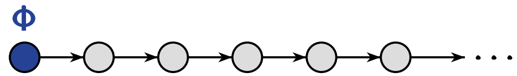
state formula



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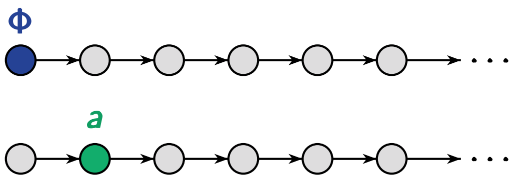
state formula

next $\bigcirc a$



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state formula

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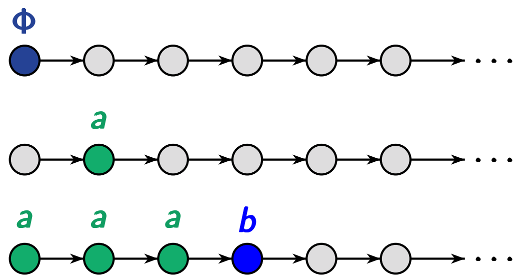
until $a \,\mathsf{U}\, b$



Derived path operators: eventually, always

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syntax of path formulas:

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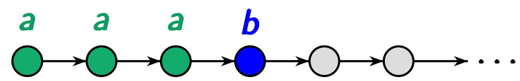
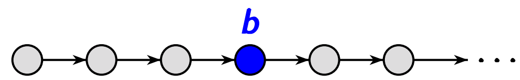


eventually

$\Diamond b \stackrel{\text{def}}{=} \mathit{true} \, \mathbf{U} \, b$



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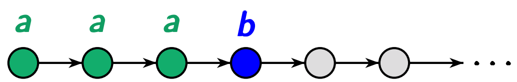

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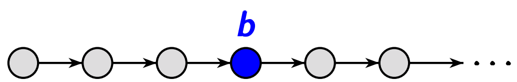
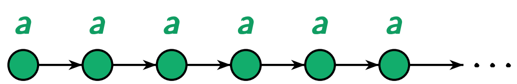


always

$\Box a \stackrel{\text{def}}{=} \neg\Diamond\neg a$



Semantics of PCTL*

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Let $\mathcal{M} = (S, P, AP, L)$ be a Markov chain.

Define by structural induction:

- a satisfaction relation $\models$ for states $s \in S$ and **PCTL*** state formulas
- a satisfaction relation $\models$ for infinite paths π in $\mathcal{M}$ and **PCTL*** path formulas

Semantics of PCTL*

$s \models \mathit{true}$

$s \models a$ iff $a \in L(s)$

$s \models \neg\Phi$ iff $s \not\models \Phi$

$s \models \Phi_1 \wedge \Phi_2$ iff $s \models \Phi_1$ and $s \models \Phi_2$

$s \models \mathbb{P}_I(\varphi)$ iff $Pr^{\mathcal{M}}(s, \varphi) \in I$

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$s \models \mathbb{P}_I(\varphi)$ iff $Pr^{\mathcal{M}}(s, \varphi) \in I$

↑ probability measure of the set of paths π with $\pi \models \varphi$

when s is viewed as the unique starting state

Semantics of PCTL* path formulas

let $\pi = s_0\, s_1\, s_2\, s_3 \ldots$ be an infinite path in $\mathcal{M}$

Semantics of PCTL* path formulas

let $\pi = s_0\, s_1\, s_2\, s_3 \ldots$ be an infinite path in $\mathcal{M}$

$$
\begin{array}{lll}
\pi \models \Phi & \text{iff} & s_0 \models \Phi \\
\pi \models \neg\varphi & \text{iff} & \pi \not\models \varphi \\
\pi \models \varphi_1 \wedge \varphi_2 & \text{iff} & \pi \models \varphi_1 \text{ and } \pi \models \varphi_2 \\
\pi \models \bigcirc\varphi & \text{iff} & s_1\, s_2\, s_3 \ldots \models \varphi \\
\pi \models \varphi_1 \,\mathsf{U}\, \varphi_2 & \text{iff} & \text{there exists } \ell \geqslant 0 \text{ such that} \\
 & & s_\ell\, s_{\ell+1}\, s_{\ell+2} \ldots \models \varphi_2 \\
 & & s_i\, s_{i+1}\, s_{i+2} \ldots \models \varphi_1 \quad \text{for } 0 \leqslant i < \ell
\end{array}
$$

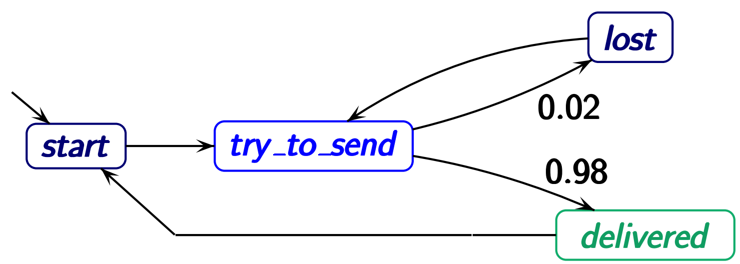
Examples for PCTL*-specifications

Examples for PCTL*-specifications

communication protocol:

$$\mathbb{P}_{=1}\big(\, \Box(\, \textit{try\_to\_send} \;\longrightarrow\; \mathbb{P}_{\geqslant 0.9}(\bigcirc \textit{delivered})\,)\,\big)$$

$$\mathbb{P}_{=1}\big(\, \Box(\, \textit{try\_to\_send} \;\longrightarrow\; \neg \textit{start} \,\mathsf{U}\, \textit{delivered}\,)\,\big)$$



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$$\mathbb{P}_{=1}\big(\, \Box(\, \textit{try\_to\_send} \;\longrightarrow\; \neg \textit{start} \,\mathsf{U}\, \textit{delivered}\,)\,\big)$$

leader election protocol for n processes:

$$\mathbb{P}_{=1}\big(\, \Diamond\, \textit{leader\_elected}\,\big)$$

$$\mathbb{P}_{\geqslant 0.9}\big(\, \bigvee_{i \leqslant n} \bigcirc^i \textit{leader\_elected}\,\big)$$

PCTL* model checking for DTMC

PCTL-80

PCTL* model checking for DTMC

PCTL-80

given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
PCTL* state formula Φ

task: check whether $s_0 \models \Phi$

PCTL* model checking for DTMC

PCTL-80

given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
PCTL* state formula Φ

task: check whether $s_0 \models \Phi$

main procedure as for CTL*:

recursively compute the satisfaction sets

$$Sat(\Psi) = \{ s \in S : s \models \Psi \}$$

for all sub-state formulas Ψ of Φ

Recursive computation of the satisfaction sets

PCTL-85

Recursive computation of the satisfaction sets

$$\begin{aligned} Sat(true) &= S \quad \text{state space of } \mathcal{M} \\ Sat(a) &= \{s \in S : a \in L(s)\} \\ Sat(\Phi_1 \wedge \Phi_2) &= Sat(\Phi_1) \cap Sat(\Phi_2) \\ Sat(\neg\Phi) &= S \setminus Sat(\Phi) \end{aligned}$$

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special case: $\varphi = \Diamond\Phi$

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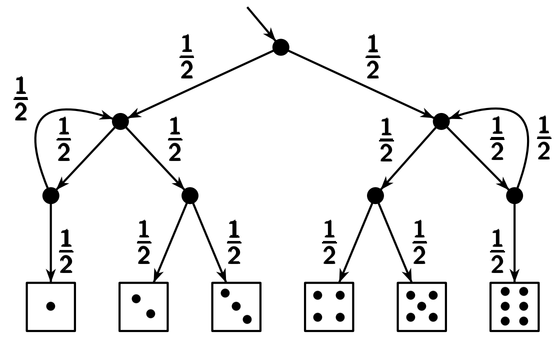
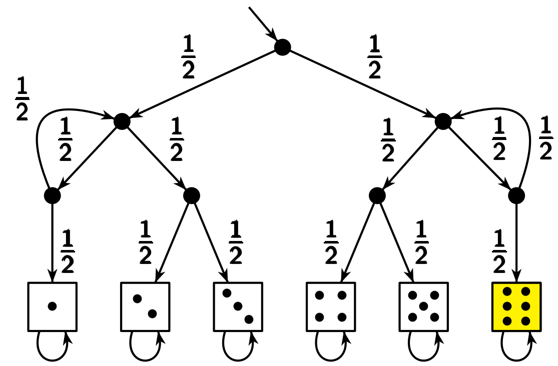
special case: $\varphi = \Diamond\Phi$

1. compute recursively $Sat(\Phi)$
2. compute $x_s = \Pr^{\mathcal{M}}(s, \Diamond\Phi)$ by solving a linear equation system

Simulating a dice by a coin [Knuth]



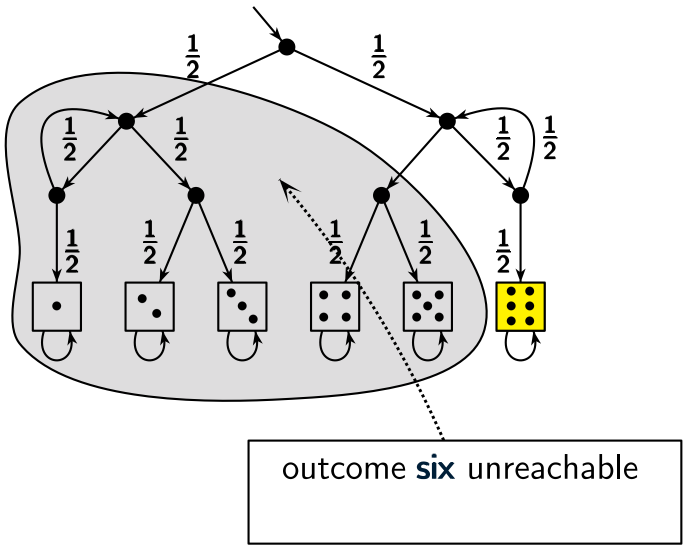
Simulating a dice by a coin [Knuth]



probability for the outcome **six**

$$\Pr^{\mathcal{M}}(\lozenge\, \mathsf{six}\,) = \textbf{?}$$

Simulating a dice by a coin [Knuth]



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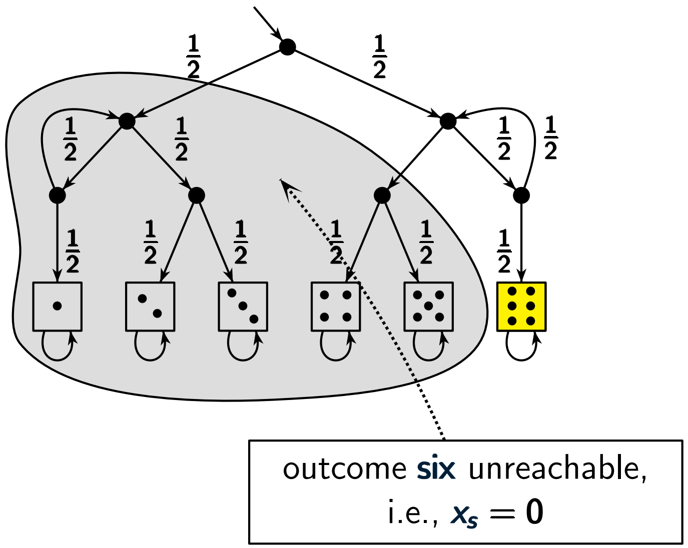
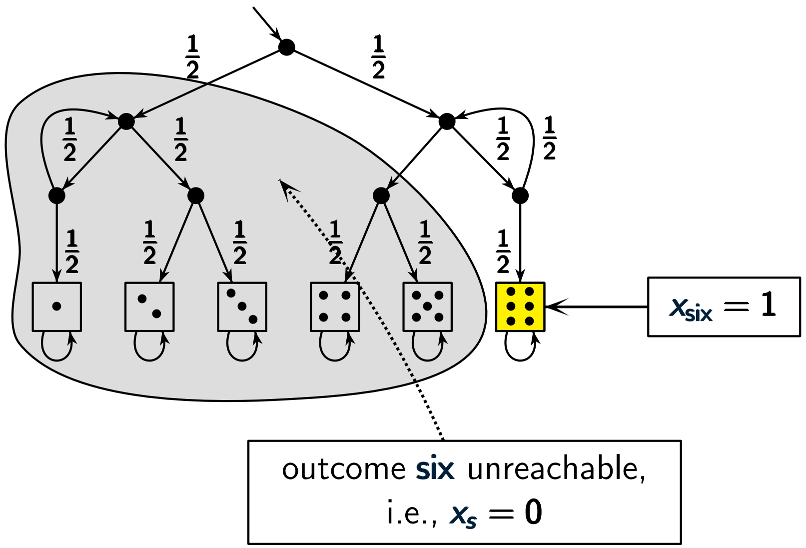


Simulating a dice by a coin [Knuth] PCTL-90



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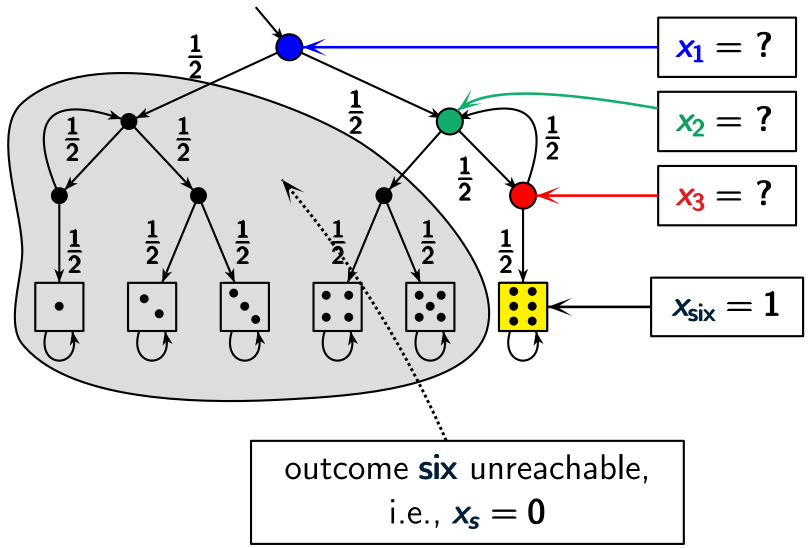

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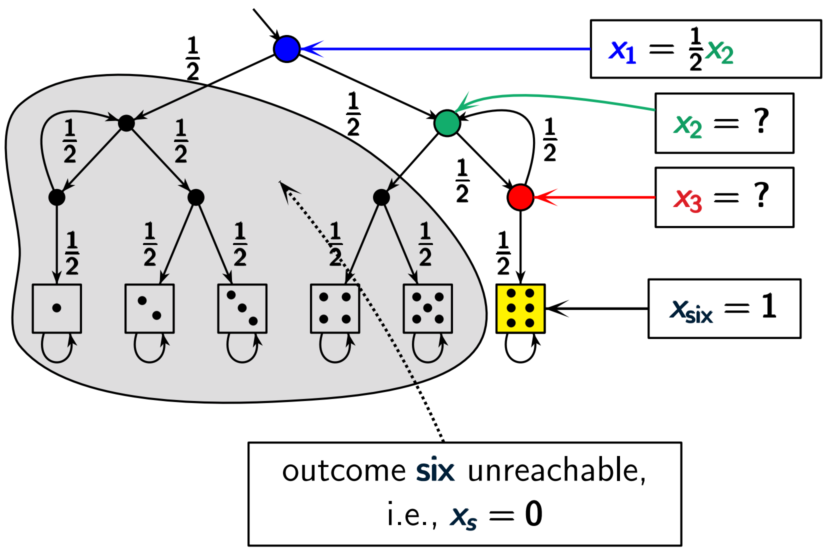

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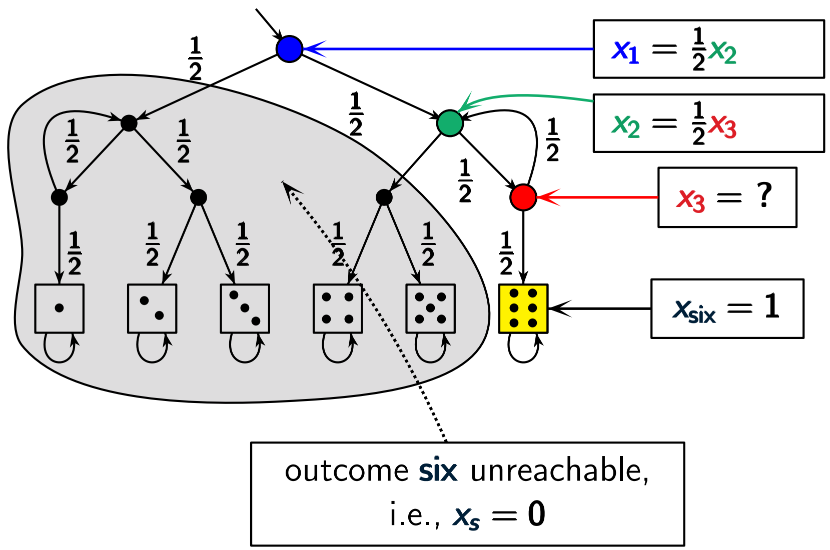

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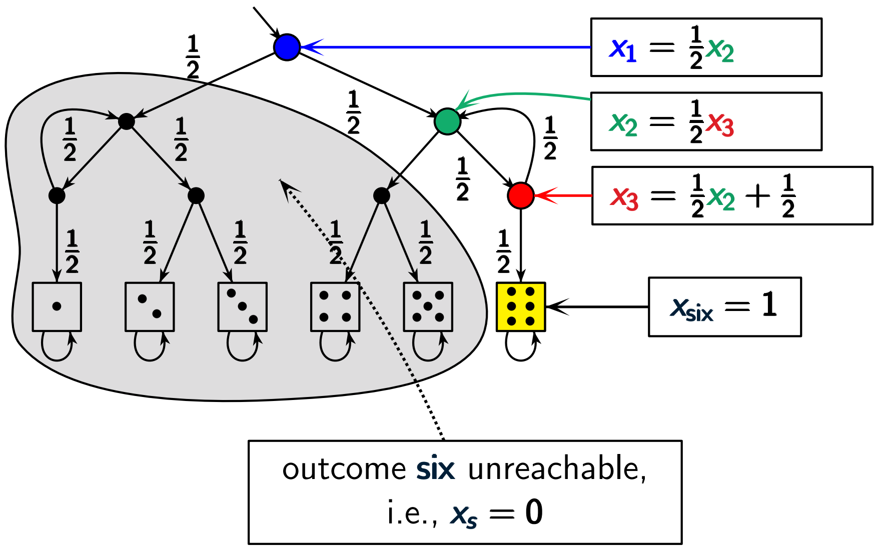

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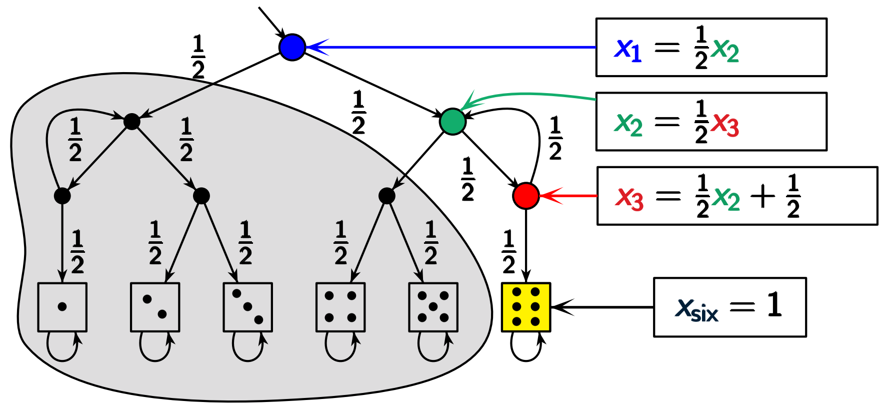
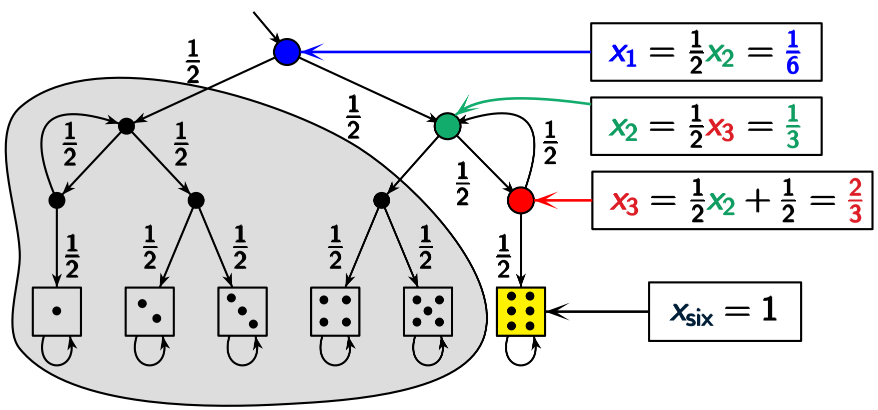


$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

Simulating a dice by a coin [Knuth] PCTL-90



$Pr^{\mathcal{M}}(\Diamond \text{ six })$
$= x_1 = \frac{1}{6}$

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

Computing reachability probabilities PCTL-100

Computing reachability probabilities

given: DTMC $\mathcal{M} = (S, P, \ldots)$ and $T \subseteq S$

task: compute $x_s = \Pr^{\mathcal{M}}(s, \Diamond T)$ for all $s \in S$

Computing reachability probabilities

given: DTMC $\mathcal{M} = (S, P, \ldots)$ and $T \subseteq S$

task: compute $x_s = \Pr^{\mathcal{M}}(s, \Diamond T)$ for all $s \in S$

1. compute S^0 and S^1

   $$S^0 = \{s \in S : x_s = 0\}$$
   $$S^1 = \{s \in S : x_s = 1\}$$

2. ...

Computing reachability probabilities

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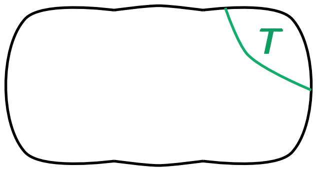

state space S

Computing reachability probabilities

given: DTMC $\mathcal{M} = (S, P, \ldots)$ and $T \subseteq S$

task: compute $x_s = \Pr^{\mathcal{M}}(s, \Diamond T)$ for all $s \in S$

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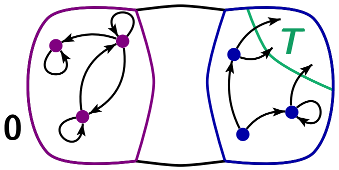
2. ...

$x_s = 0$



state space S

$x_s = 1$

Computing reachability probabilities

given: DTMC $\mathcal{M} = (S, P, \dots)$ and $T \subseteq S$

task: compute $x_s = Pr^{\mathcal{M}}(s, \Diamond T)$ for all $s \in S$

1. compute S^0 and S^1

$$S^0 = \{s \in S : x_s = 0\} = \{s : s \not\models \exists \Diamond T\}$$
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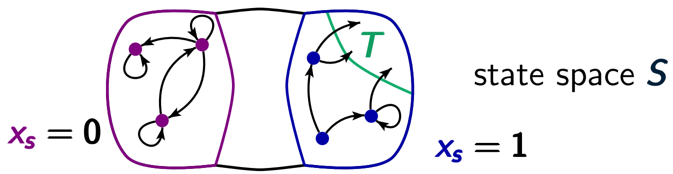
2. …



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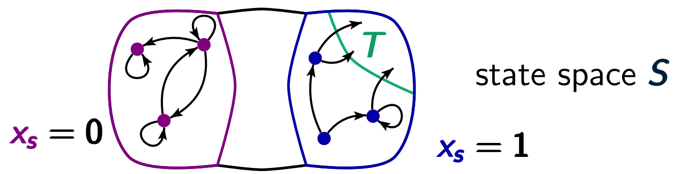

Computing reachability probabilities

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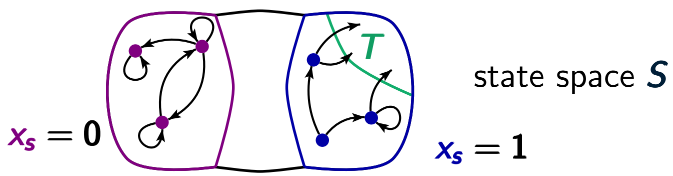

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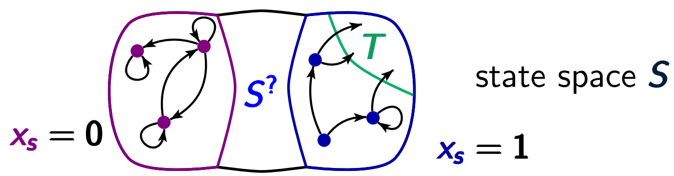
2. compute x_s for $s \in S^? = S \setminus (S^0 \cup S^1)$



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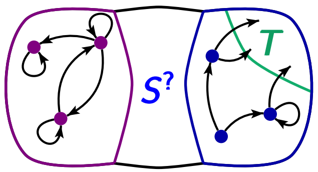
2. compute x_s for $s \in S^? = \{s : 0 < x_s < 1\}$

$x_s = 0$



state space S

$x_s = 1$

Computing reachability probabilities

given: DTMC $\mathcal{M} = (S, P, \ldots)$ and $T \subseteq S$

task: compute $x_s = \Pr^{\mathcal{M}}(s, \Diamond T)$ for all $s \in S$

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$$S^0 = \{s \in S : x_s = 0\} = \{s : s \not\models \exists \Diamond T\}$$
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2. compute x_s for $s \in S^? = \{s : 0 < x_s < 1\}$

by solving a linear equation system

Computing reachability probabilities

task: compute $x_s = \Pr^{\mathcal{M}}(s, \Diamond T)$ for all $s \in S^?$

by solving the equation system:

$$x_s = \sum_{s' \in S^?} P(s, s') \cdot x_{s'} + P(s, S^1)$$

$$P(s, S^1) = \sum_{u \in S^1} P(s, u)$$

Computing reachability probabilities

task: compute $x_s = \Pr^{\mathcal{M}}(s, \Diamond T)$ for all $s \in S^?$

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$$x_s = \sum_{s' \in S^?} P(s, s') \cdot x_{s'} + P(s, S^1)$$

probability for paths of the form

$s \, \underbrace{u_1 \, u_2 \ldots u_k}_{u_j \in S^1} \, t$ with $t \in T$

Computing reachability probabilities

task: compute $x_s = \Pr^{\mathcal{M}}(s, \Diamond T)$ for all $s \in S^?$

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probability for paths of the form

$$\underbrace{s\ s_1\ s_2 \ldots s_m}_{s_i \in S^?}\ \underbrace{u_1\ u_2 \ldots u_k}_{u_j \in S^1}\ t \quad \text{with } t \in T$$

Computing reachability probabilities

task: compute $x_s = \Pr^{\mathcal{M}}(s, \Diamond T)$ for all $s \in S^?$

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$$x_s = \sum_{s' \in S^?} P(s,s') \cdot x_{s'} + P(s, S^1)$$

$$x = A \cdot x + b$$

Computing reachability probabilities

task: compute $x_s = \Pr^{\mathcal{M}}(s, \Diamond T)$ for all $s \in S^?$

by solving the equation system:

$$x_s = \sum_{s' \in S^?} P(s,s') \cdot x_{s'} + P(s, S^1)$$

$$x = A \cdot x + b$$

matrix $A = \big(P(s,s')\big)_{s,s' \in S^?}$

vectors $x = \big(x_s\big)_{s \in S^?}$

$b = \big(P(s,S^1)\big)_{s \in S^?}$

Computing reachability probabilities

task: compute $x_s = \Pr^{\mathcal{M}}(s, \Diamond T)$ for all $s \in S^?$

by solving the equation system:

$$x_s = \sum_{s' \in S^?} P(s,s') \cdot x_{s'} + P(s, S^1)$$

$$x = A \cdot x + b$$

iff

$$(\mathbf{I} - A) \cdot x = b$$

matrix $A = \big(P(s,s')\big)_{s,s' \in S^?}$

vectors $x = \big(x_s\big)_{s \in S^?}$

$b = \big(P(s,S^1)\big)_{s \in S^?}$

identity matrix $\mathbf{I}$

Computing reachability probabilities

task: compute $x_s = \Pr^{\mathcal{M}}(s, \lozenge T)$ for all $s \in S^?$

by solving the equation system:

$$x_s \;=\; \sum_{s' \in S^?} P(s,s') \cdot x_{s'} \;+\; P(s, S^1)$$

$$x = A \cdot x + b \quad \text{iff} \quad (\mathbf{I} - A) \cdot x = b$$

linear equation system with non-singular matrix $\mathbf{I} - A$

Computing reachability probabilities

task: compute $x_s = \Pr^{\mathcal{M}}(s, \lozenge T)$ for all $s \in S^?$

by solving the equation system:

$$x_s \;=\; \sum_{s' \in S^?} P(s,s') \cdot x_{s'} \;+\; P(s, S^1)$$

$$x = A \cdot x + b \quad \text{iff} \quad (\mathbf{I} - A) \cdot x = b$$

linear equation system with non-singular matrix $\mathbf{I} - A$

$\Downarrow$

unique solution

PCTL

sublogic of **PCTL*** where only path formulas of the form $\bigcirc\Phi$ and $\Phi_1 \,\mathsf{U}\, \Phi_2$ are allowed

PCTL

sublogic of **PCTL*** where only path formulas of the form $\bigcirc\Phi$ and $\Phi_1 \,\mathsf{U}\, \Phi_2$ are allowed

state formulas:

$$\Phi \;::=\; \mathit{true} \;\big|\; a \;\big|\; \Phi_1 \wedge \Phi_2 \;\big|\; \neg\Phi \;\big|\; \mathbb{P}_I(\varphi)$$

path formulas:

$$\varphi \;::=\; \bigcirc\Phi \;\big|\; \Phi_1 \,\mathsf{U}\, \Phi_2$$

PCTL

sublogic of **PCTL*** where only path formulas of the form $\bigcirc\Phi$ and $\Phi_1 \,\mathsf{U}\, \Phi_2$ are allowed

state formulas:

$$\Phi \;::=\; true \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \mathbb{P}_I(\varphi)$$

path formulas:

$$\varphi \;::=\; \bigcirc\Phi \mid \Phi_1 \,\mathsf{U}\, \Phi_2 \mid \Diamond\Phi \mid \Box\Phi$$

$$\mathbb{P}_I(\Diamond\Phi) \stackrel{\text{def}}{=} \mathbb{P}_I(true \,\mathsf{U}\, \Phi)$$

PCTL
PCTL-150

sublogic of **PCTL*** where only path formulas of the form $\bigcirc\Phi$ and $\Phi_1 \,\mathsf{U}\, \Phi_2$ are allowed

state formulas:

$$\Phi \;::=\; true \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \mathbb{P}_I(\varphi)$$

path formulas:

$$\varphi \;::=\; \bigcirc\Phi \mid \Phi_1 \,\mathsf{U}\, \Phi_2 \mid \Diamond\Phi \mid \Box\Phi$$

$$\mathbb{P}_I(\Diamond\Phi) \stackrel{\text{def}}{=} \mathbb{P}_I(true \,\mathsf{U}\, \Phi)$$

e.g., $\mathbb{P}_{<0.4}(\Box\Phi) \stackrel{\text{def}}{=} \mathbb{P}_{>0.6}(\Diamond\neg\Phi)$

note: $\Pr^{\mathcal{M}}(s, \Box\Phi) = 1 - \Pr^{\mathcal{M}}(s, \Diamond\neg\Phi)$

PCTL model checking

given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
PCTL formula Φ

task: check whether $s_0 \models \Phi$

recursive computation of $Sat(\Psi) = \{s \in S : s \models \Psi\}$ for all subformulas Ψ of Φ

- treatment of the propositional logic fragment: $\surd$
- treatment of the probability operator $\mathbb{P}_I(\varphi)$

  graph algorithms + matrix/vector operations

  ↑

  next: matrix/vector multiplication
  until: linear equation system

PCTL model checking
PCTL-170

given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
PCTL formula Φ

task: check whether $s_0 \models \Phi$

recursive computation of $Sat(\Psi) = \{s \in S : s \models \Psi\}$ for all subformulas Ψ of Φ

- treatment of the propositional logic fragment: $\surd$
- treatment of the probability operator $\mathbb{P}_I(\varphi)$

  graph algorithms + matrix/vector operations

time complexity: $\mathcal{O}(\,\mathbf{poly}(\mathcal{M}) \cdot |\Phi|\,)$

PCTL* model checking

given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
PCTL*-state formula Φ
task: check whether $s_0 \models \Phi$

recursive computation of $Sat(\Psi) = \{s \in S : s \models \Psi\}$
for all sub-state formulas Ψ of Φ

- treatment of the propositional logic fragment: $\surd$
- treatment of the probability operator $\mathbb{P}_I(\varphi)$

PCTL* path formula $\varphi \rightsquigarrow$ LTL formula φ'
↑
path formula without
state formulas $\mathbb{P}_I(\ldots)$

PCTL* model checking

given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
PCTL*-state formula Φ
task: check whether $s_0 \models \Phi$

recursive computation of $Sat(\Psi) = \{s \in S : s \models \Psi\}$
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- treatment of the propositional logic fragment: $\surd$
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PCTL* path formula $\varphi \rightsquigarrow$ LTL formula φ'

... automata-based approach for φ' ...

PCTL* model checking

given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
PCTL*-state formula Φ
task: check whether $s_0 \models \Phi$

treatment of the probability operator $\mathbb{P}_I(\varphi)$

PCTL* path formula $\varphi \rightsquigarrow$ LTL formula φ'

by replacing each maximal state-subformula
of φ with a fresh atomic proposition

PCTL* model checking

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PCTL* path formula $\varphi \rightsquigarrow$ LTL formula φ'

by replacing each maximal state-subformula
of φ with a fresh atomic proposition

$$\Diamond\big(a \,\mathsf{U}\, \mathbb{P}_{\geqslant 0.7}(\Box\Diamond b) \ \wedge \Box\ \mathbb{P}_{<0.3}(\bigcirc\Box c)\ \big)$$

PCTL* model checking

given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
PCTL*-state formula Φ
task: check whether $s_0 \models \Phi$

treatment of the probability operator $\mathbb{P}_I(\varphi)$

PCTL* path formula $\varphi \rightsquigarrow$ LTL formula φ'

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$$\Diamond(a \, \mathsf{U} \, \mathbb{P}_{\geqslant 0.7}(\Box\Diamond b) \wedge \Box \, \mathbb{P}_{<0.3}(\bigcirc\Box c))$$

PCTL* model checking

given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
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task: check whether $s_0 \models \Phi$

treatment of the probability operator $\mathbb{P}_I(\varphi)$

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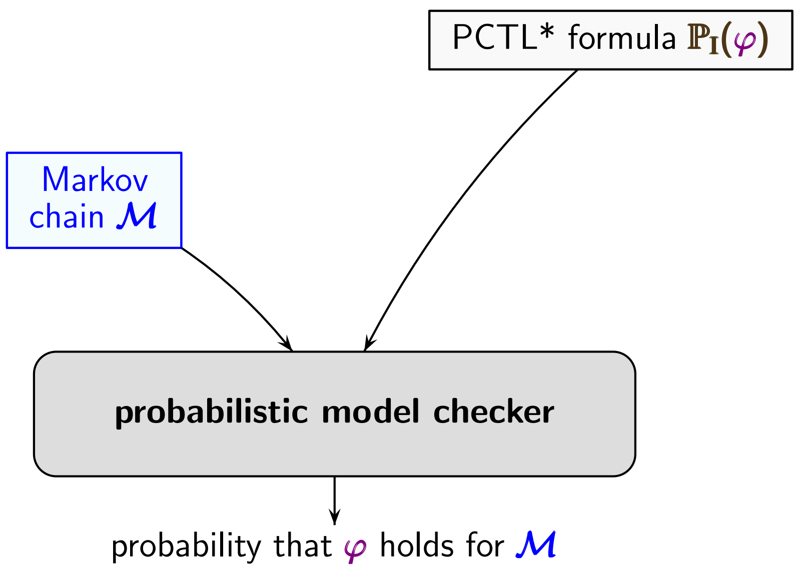
by replacing each maximal state-subformula of φ with a fresh atomic proposition

$$\Diamond(a \, \mathsf{U} \, \mathbb{P}_{\geqslant 0.7}(\Box\Diamond b) \wedge \Box \, \mathbb{P}_{<0.3}(\bigcirc\Box c))$$

$$\Downarrow$$

$$\Diamond(a \, \mathsf{U} \, d \wedge \Box e)$$

PCTL* model checking



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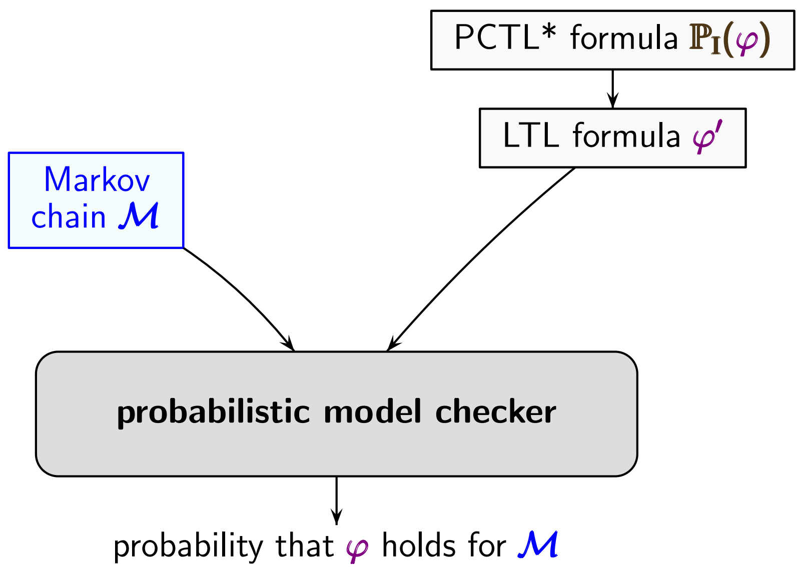

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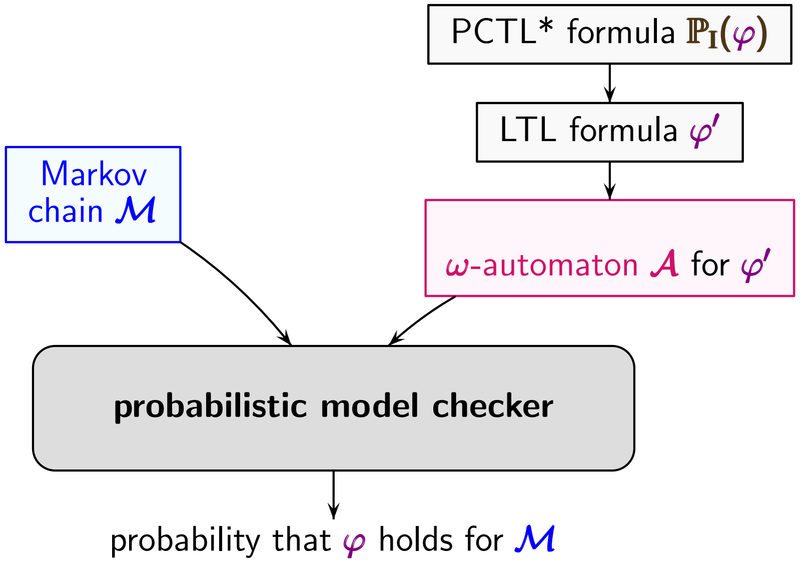

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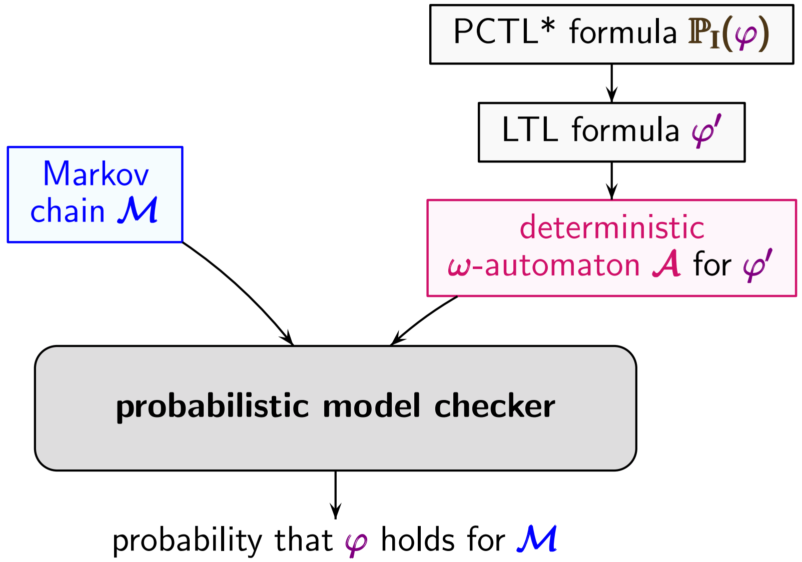

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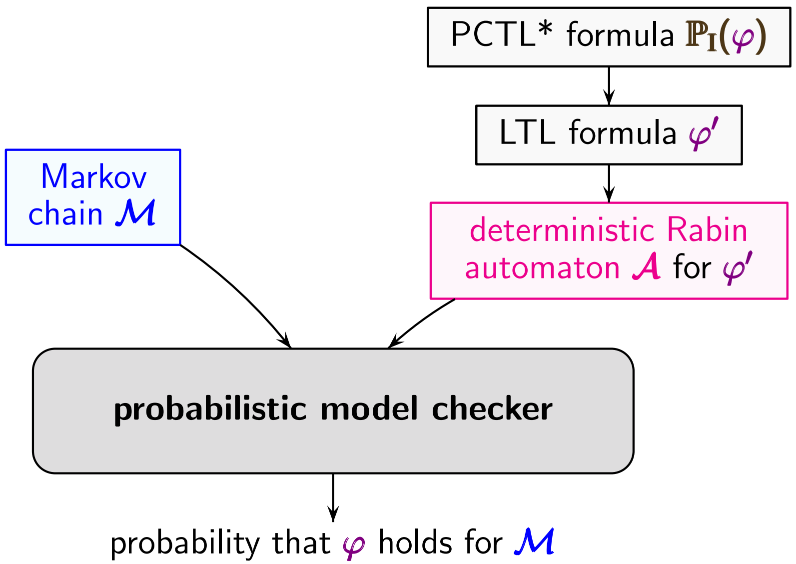

Deterministic Rabin automata (DRA)

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semantics of the acceptance condition:

$$\bigvee_{1 \leqslant i \leqslant k} (\Diamond\Box\neg L_i \ \wedge \ \Box\Diamond U_i)$$

Accepted language of a DRA

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, Acc)$ be an DRA where

$$Acc = \{(L_1, U_1), \ldots, (L_k, U_k)\} \qquad L_i, U_i \subseteq Q$$

accepted language:

$$\mathcal{L}_\omega(\mathcal{A}) = \{ \sigma \in \Sigma^\omega : \text{the run for } \sigma \text{ in } \mathcal{A} \text{ fulfills } Acc \}$$

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PCTL-355

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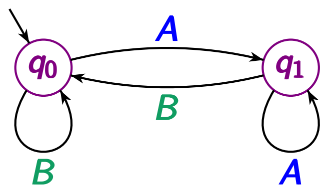
Let $\rho = q_0\, q_1\, q_2 \ldots$ is the run for some infinite word σ.

ρ fulfills Acc iff

$$\exists i \in \{1, \ldots, k\}.\ \inf(\rho) \cap L_i = \varnothing \ \wedge\ \inf(\rho) \cap U_i \neq \varnothing$$

where $\inf(\rho) = \{q \in Q : \overset{\infty}{\exists} \ell \in \mathbb{N}.\ q = q_\ell\}$

Example: DRA



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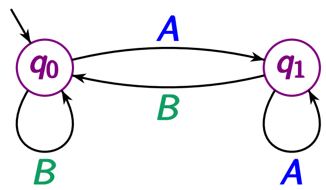
Example: DRA

PCTL-360



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$\Diamond\Box$ "eventually forever"

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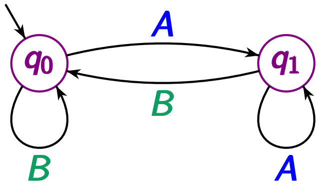

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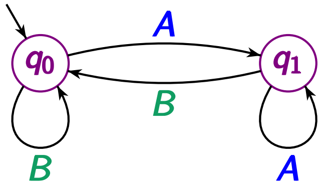

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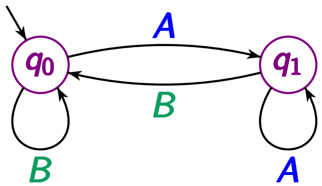

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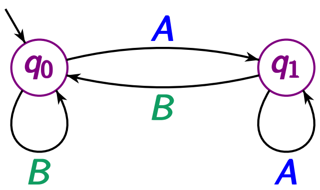

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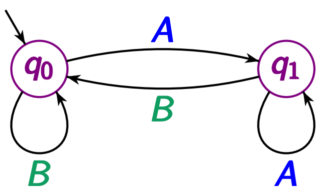

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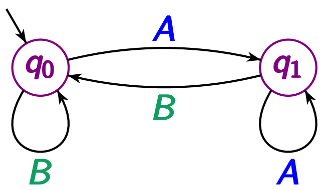

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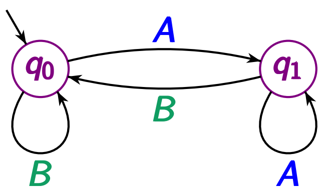

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Fundamental result: LTL-2-DRA

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For each LTL formula φ over AP there exists a DRA $\mathcal{A}$ with the alphabet $\Sigma = 2^{AP}$ s.t.

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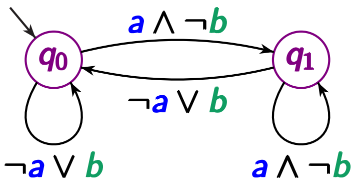

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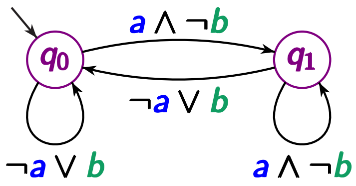

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LTL formula $\Diamond\Box(a \wedge \neg b)$

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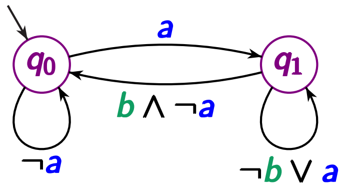

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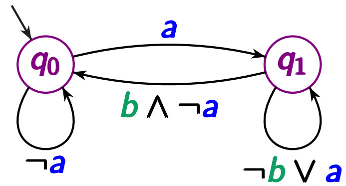

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$\Diamond\Box\neg q_1 \;\wedge\; \Box\Diamond q_0$

LTL formula
$\Box(a \rightarrow \Diamond(b \wedge \neg a)) \;\wedge\; \Diamond\Box\neg a$

PCTL* model checking

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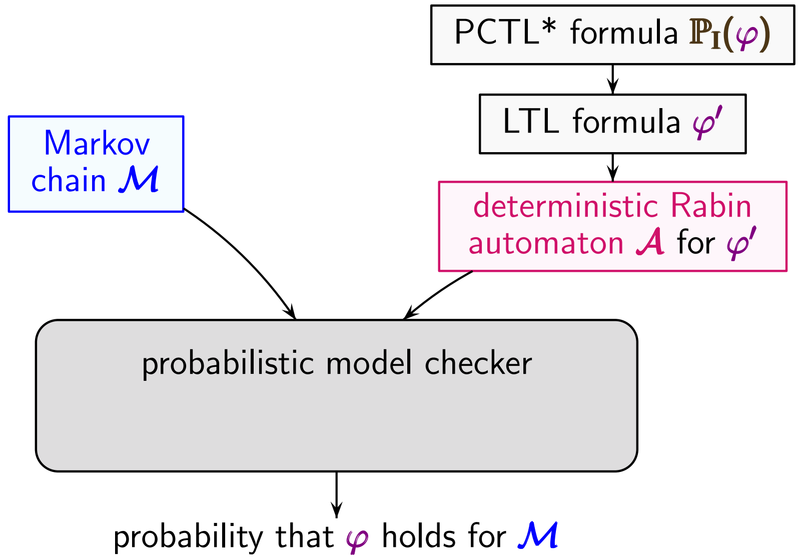

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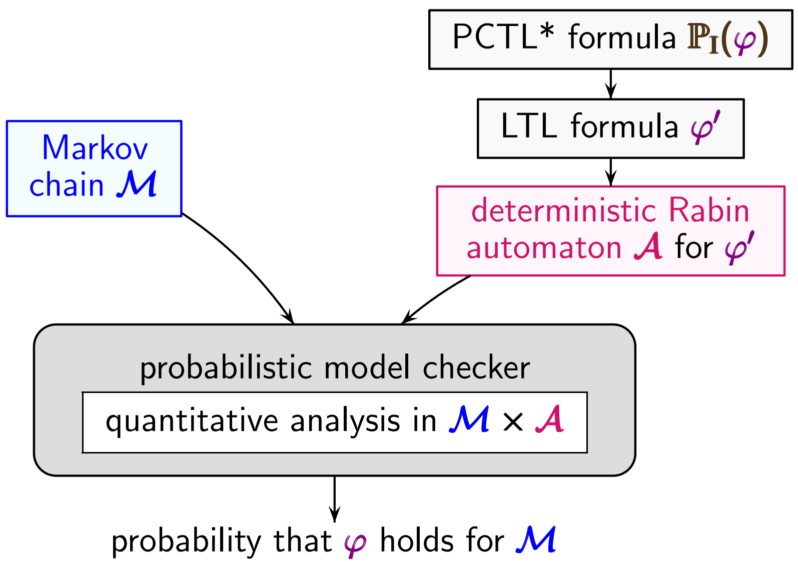

Product of a Markov chain and a DRA

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given: Markov chain $\mathcal{M} = (S, P, AP, L)$
DRA $\mathcal{A} = (Q, 2^{AP}, \delta, q_0, Acc)$
goal: define a Markov chain $\mathcal{M} \times \mathcal{A}$

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$$\Pr^{\mathcal{M}}(s, \mathcal{A}) = \Pr^{\mathcal{M}}\{\pi \in Paths(s) : trace(\pi) \in \mathcal{L}_\omega(\mathcal{A})\}$$

can be derived by a probabilistic reachability analysis in the product-chain $\mathcal{M} \times \mathcal{A}$

$$trace(s_0\, s_1\, s_2 \ldots) \;=\; L(s_0)\, L(s_1)\, L(s_2) \ldots \in \left(2^{AP}\right)^\omega$$

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idea: define a Markov chain $\mathcal{M} \times \mathcal{A}$ s.t. ...

path π in $\mathcal{M}$

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run for $trace(\pi)$ in $\mathcal{A}$

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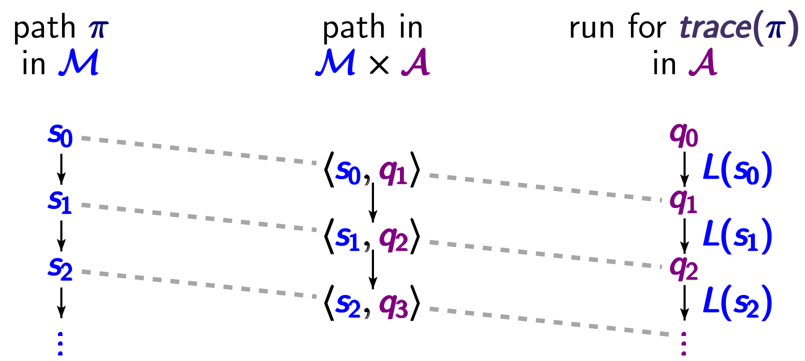

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successor state in $\mathcal{A}$ of the initial DRA-state q_0 for the input symbol $L(s) \in 2^{AP}$

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probability measure of all paths π in the product s.t. $\pi|_{\mathcal{A}}$ satisfies the acceptance condition of $\mathcal{A}$

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$$Pr^{\mathcal{M}}(s, \mathcal{A}) = Pr^{\mathcal{M}\times\mathcal{A}}\big(\langle s, q_s \rangle, \Diamond\, accBSCC \big)$$

↑ union of accepting BSCCs in $\mathcal{M} \times \mathcal{A}$ i.e., BSCC C s.t.

$$\exists i \in \{1, \ldots, k\}.\ \ C \cap L_i = \varnothing \ \wedge\ C \cap U_i \neq \varnothing$$

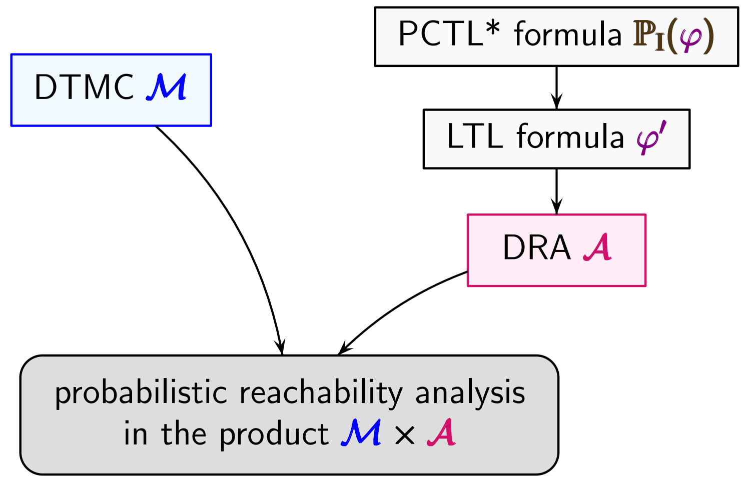
Summary: PCTL* model checking

given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
PCTL* state formula Φ
task: check whether $\mathcal{M} \models \Phi$

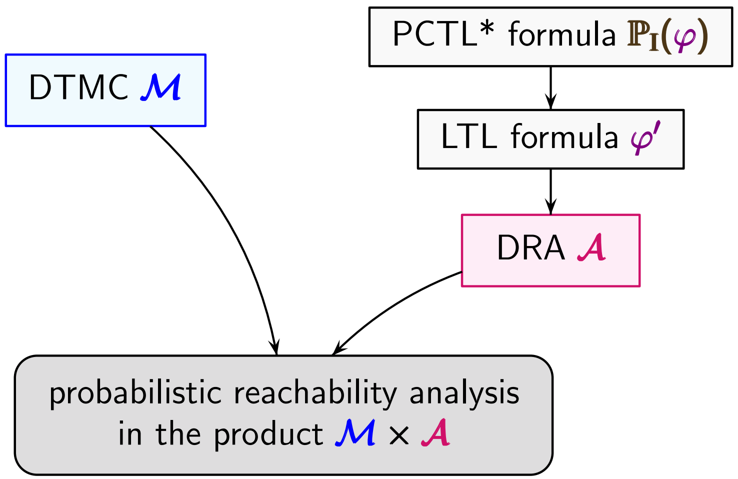
method: bottom-up treatment of sub-state formulas Ψ to compute

$$Sat(\Psi) = \{ s \in S : s \models \Psi \}$$

- propositional logic fragment: obvious
- probability operator $\mathbb{P}_I(\varphi)$ via
  - construction of a DRA $\mathcal{A}$ for φ
  - probabilistic reachability analysis in $\mathcal{M} \times \mathcal{A}$

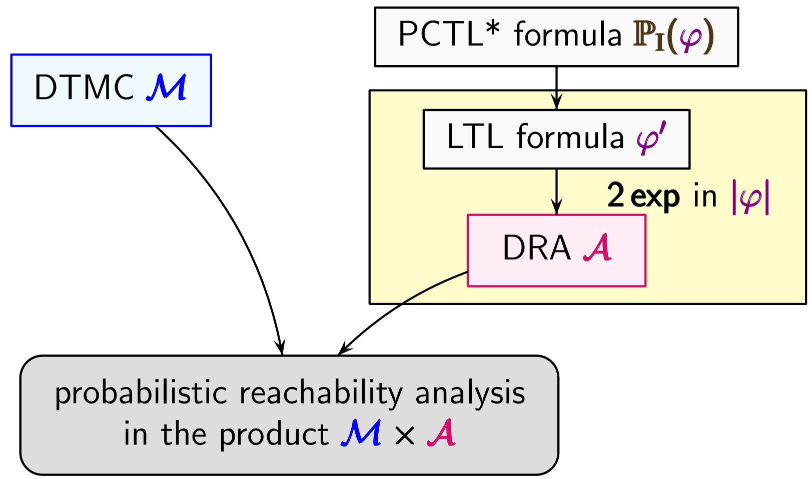


1. graph analysis to compute the accepting BSCCs of the product
2. linear equation system for the probabilities to reach an accepting BSCC



1. graph analysis to compute the accepting BSCCs of the product
2. linear equation system

time complexity: polynomial in the sizes of $\mathcal{M}$ and $\mathcal{A}$



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Discrete-time Markovian models

Part 1: discrete-time Markov chains (DTMC)

1. basic definitions
2. probabilistic computation tree logic (PCTL/PCTL*)
3. expected rewards ⟵

Part 2: Markov decision processes (MDP)

1. basic definitions
2. PCTL/PCTL* model checking
3. fairness

Markov reward model (MRM)

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Markov chain $\mathcal{M} = (S, P, AP, L, rew)$ with a reward function for the states:

$$rew : S \to \mathbb{N}$$

analogously: rewards for edges given by $rew : S \times S \to \mathbb{N}$

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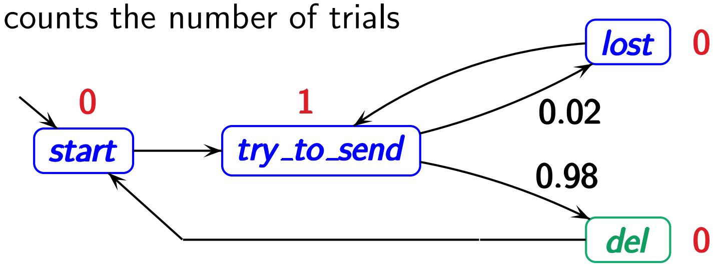
formalization by accumulated rewards of finite paths

$$rew(s_0\, s_1 \ldots s_n) \;=\; \sum_{0 \leqslant i < n} rew(s_i)$$

analogously: rewards for edges given by $rew : S \times S \to \mathbb{N}$

Example: Markov reward model

communication protocol with reward function that counts the number of trials



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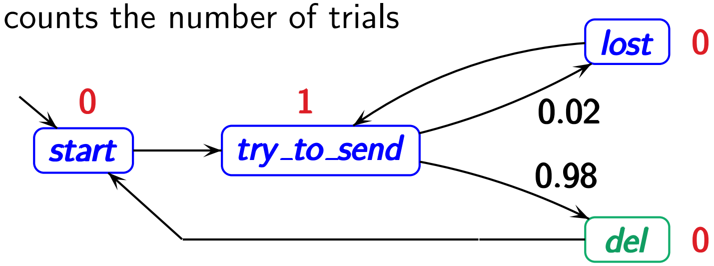

accumulated reward of finite paths, e.g.,

$$rew(start\ try\ lost\ try\ del) \;=\; 2$$

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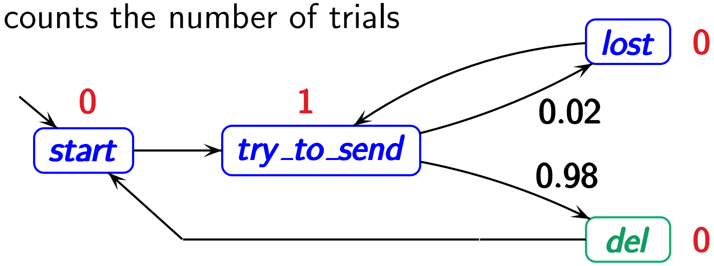

measures of interest, e.g.,

$\Pr^{\mathcal{M}}(\Diamond^{\leqslant 3} del)$ probability to deliver a message within at most **3** trials

↑ reachability with reward bound $\leqslant 3$

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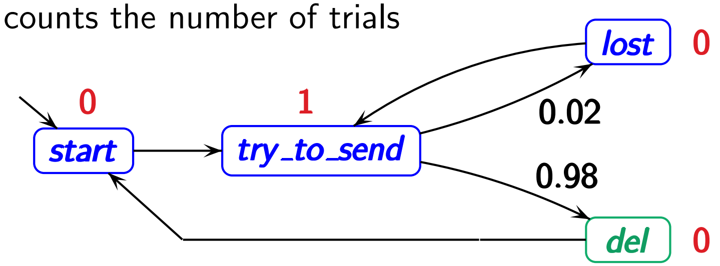

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$\mathbb{E}^{\mathcal{M}}(\Diamond^{=?} del)$ expected number of trials

Reward-based extension of PCTL

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- probability operator for reward-bounded until path formulas $\mathbb{P}_I(\Phi_1 \, \mathsf{U}^{\leqslant r} \, \Phi_2)$
- expected reward operator $\mathbb{C}_{\leqslant r}(\Phi)$

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$s \models \mathbb{C}_{\leqslant r}(\Phi)$ iff the expected accumulated reward on paths from s to a Φ-state is $\leqslant r$

Reward-based extension of PCTL

- probability operator for reward-bounded until path formulas $\mathbb{P}_I(\Phi_1 \mathsf{U}^{\leqslant r} \Phi_2)$
- expected reward operator $\mathbb{C}_{\leqslant r}(\Phi)$

$s \models \mathbb{C}_{\leqslant r}(\Phi)$ iff the expected accumulated reward on paths from s to a Φ-state is $\leqslant r$

e.g., for communication protocols:

$\mathbb{P}_{>0.9}(\Diamond^{\leqslant 3} del)$	probability for delivering the message within at most 3 trials is > 0.9
$\mathbb{C}_{\leqslant 5}(del)$	average number of trials is $\leqslant 5$

Model checking reward-based properties

Model checking reward-based properties

treatment of $\mathbb{P}_I(\Phi_1 \mathsf{U}^{\leqslant r} \Phi_2)$ where $r \in \mathbb{N}$

compute $\Pr^{\mathcal{M}}(s, \Phi_1 \mathsf{U}^{\leqslant i} \Phi_2)$ iteratively for increasing reward bound $i = 0, 1, 2, \ldots, r$

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Let $\rho(s, i) = \Pr^{\mathcal{M}}(s, \Phi_1 \mathsf{U}^{\leqslant i} \Phi_2)$. Then:

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if $s \models \exists(\Phi_1 \, \mathsf{U} \, \Phi_2) \wedge \neg\Phi_2$ and $i \geqslant rew(s)$ then

$$\rho(s,i) = \sum_{s' \in S} P(s,s') \cdot \rho(s', i - rew(s))$$

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if $s \models \Phi_2$ then: $\rho(s,i) = 1$

in all other cases: $\rho(s,i) = 0$

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compute the expected accumulated rewards
by solving the linear equation system:

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treatment of the $\mathbb{C}_{\leqslant r}(\Phi)$: assuming $Pr^{\mathcal{M}}(\Diamond\Phi) = 1$

compute the expected accumulated rewards
by solving the linear equation system:

$$x_s = rew(s) + \sum_{s' \in S} P(s,s') \cdot x_{s'} \quad \text{if } s \not\models \Phi$$

$$x_s = 0 \quad \text{if } s \models \Phi$$

Model checking reward-based properties

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treatment of the $\mathbb{C}_{\leqslant r}(\Phi)$: assuming $Pr^{\mathcal{M}}(\Diamond\Phi) = 1$

compute the expected accumulated rewards
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time complexity:

unit rewards: polynomial in $size(\mathcal{M})$ and $\log r$

expected reward: linear equation system

reward-bounded until: repeated squaring

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unit rewards: polynomial in $size(\mathcal{M})$ and $\log r$

general case: *NP*-hard [Larroussinie/Sproston]

Discrete-time Markovian models

Part 1: discrete-time Markov chains (DTMC)

1. basic definitions
2. probabilistic computation tree logic (PCTL/PCTL*)
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Part 2: Markov decision processes (MDP)

1. basic definitions
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Markov decision processes (MDP)

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extend Markov chains by nondeterminism

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- modeling asynchronous distributed systems by interleaving

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extend Markov chains by nondeterminism

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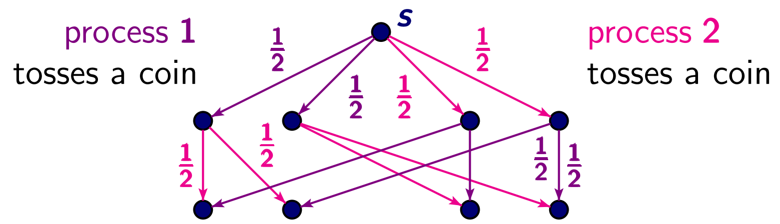
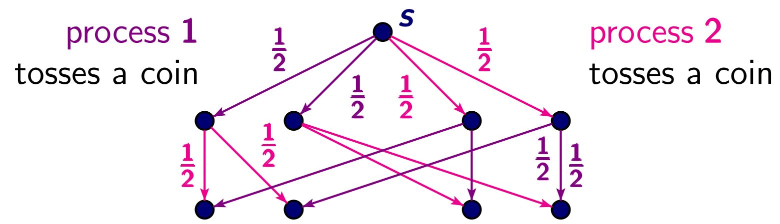


Markov decision processes (MDP)

extend Markov chains by nondeterminism

- modeling asynchronous distributed systems by interleaving
- useful for abstraction purposes
- representation of the interface with an unpredictable environment (e.g., human user)



Markov decision process (MDP)

$\mathcal{M} = (S, Act, P, AP, L)$

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- finite state space S

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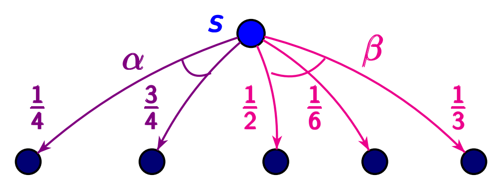
Markov decision process (MDP)

$\mathcal{M} = (S, Act, P, AP, L)$

- finite state space S
- Act finite set of actions
- $P : S \times Act \times S \to [0, 1]$ s.t.
  $\forall s \in S \;\; \forall \alpha \in Act. \;\; \sum_{s' \in S} P(s, \alpha, s') \in \{0, 1\}$



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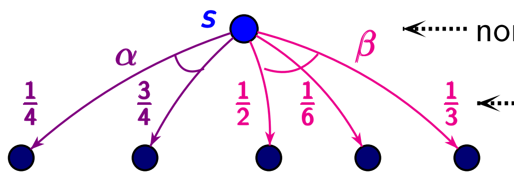

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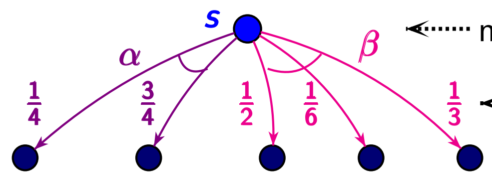
  $\forall s \in S\ \ \forall \alpha \in Act.\ \ \sum_{s' \in S} P(s,\alpha,s') \in \{0,1\}$
  $\nearrow$ $\alpha \notin Act(s)$ $\nwarrow$ $\alpha \in Act(s)$



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- $P : S \times Act \times S \to [0,1]$ s.t.
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  and $Act(s) \neq \varnothing$
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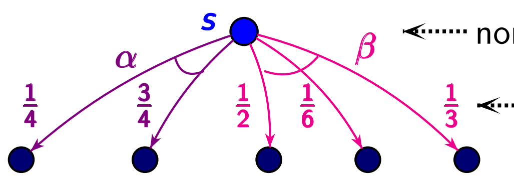

Markov decision process (MDP)

$\mathcal{M} = (S, Act, P, AP, L)$ + initial distribution, rewards, ...

- finite state space S
- Act finite set of actions
- $P : S \times Act \times S \to [0,1]$ s.t.
  $\forall s \in S\ \ \forall \alpha \in Act.\ \ \sum_{s' \in S} P(s,\alpha,s') \in \{0,1\}$
  and $Act(s) \neq \varnothing$
  $\nearrow$ $\alpha \notin Act(s)$ $\nwarrow$ $\alpha \in Act(s)$
- AP set of atomic propositions
- labeling $L : S \to 2^{AP}$

  ⋮

Randomized mutual exclusion protocol

Randomized mutual exclusion protocol
PMC-72

- 2 concurrent processes P_1, P_2 with 3 phases:

n_i	noncritical actions of process P_i
w_i	waiting phase of process P_i
c_i	critical section of process P_i

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PMC-72

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PMC-72

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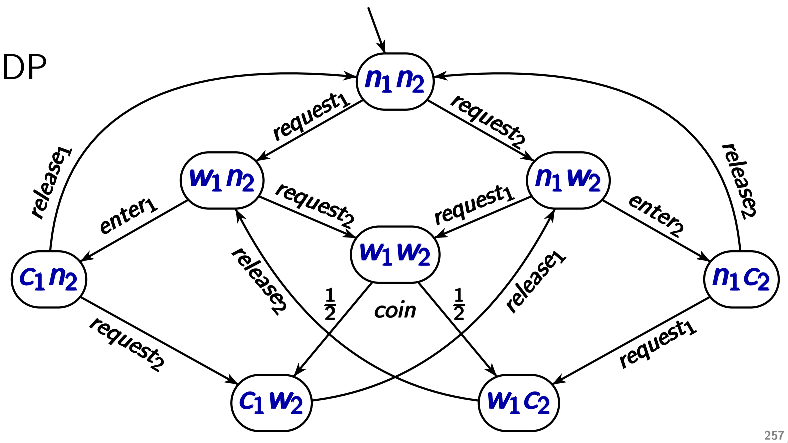
- competition of both processes are waiting

- resolved by a randomized arbiter who tosses a coin

Randomized mutual exclusion protocol

- interleaving of the request operations
- competition if both processes are waiting
- randomized arbiter tosses a coin if both are waiting

MDP



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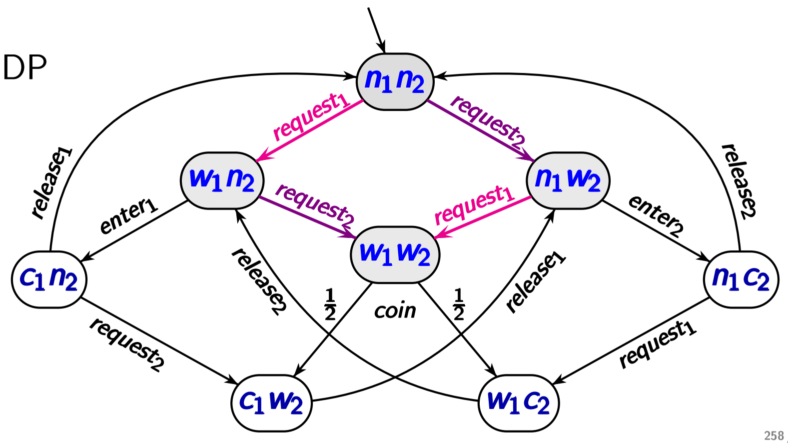

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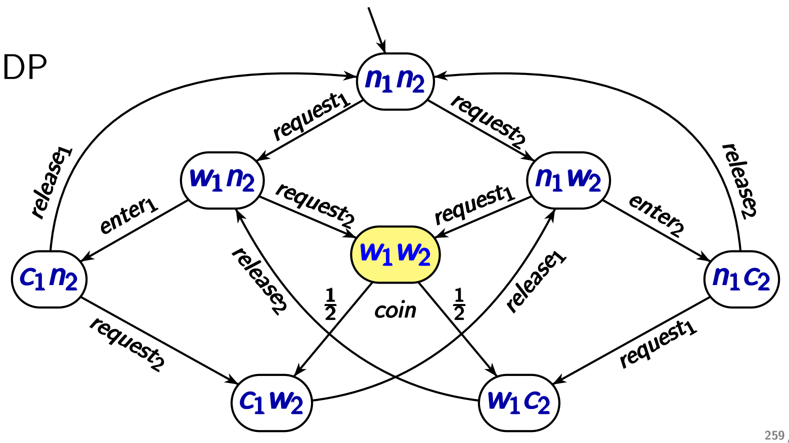

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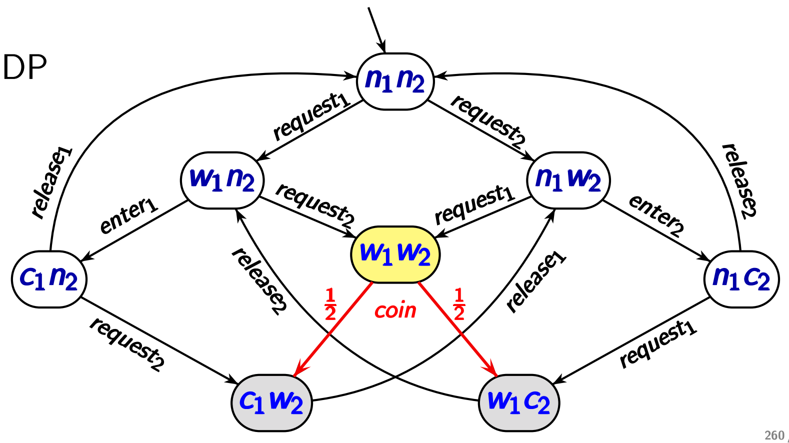

Properties of the randomized MUTEX

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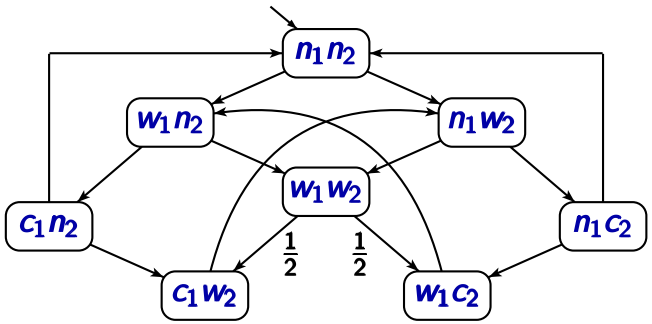
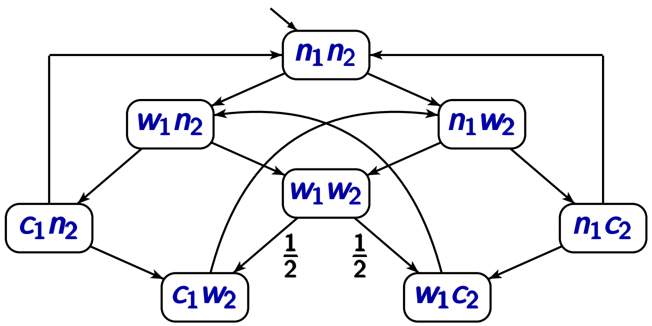


safety: the processes are never simultaneously in their critical section

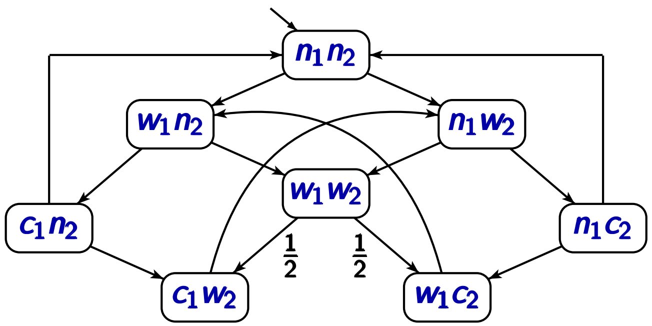
Properties of the randomized MUTEX



safety: the processes are never simultaneously in their critical section

holds on all paths as state $\langle c_1, c_2 \rangle$ is unreachable

Properties of the randomized MUTEX



liveness: each waiting process will eventually enter its critical section

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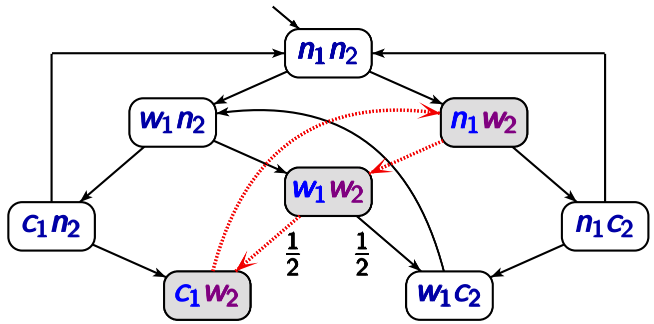
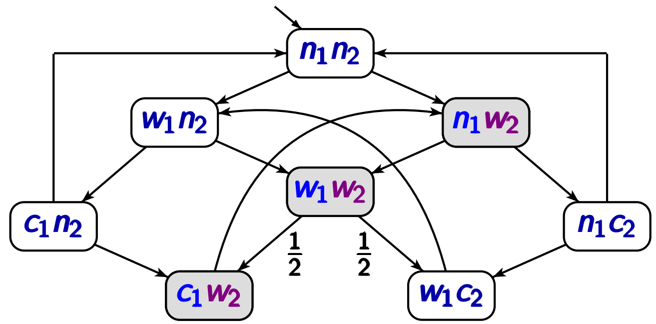


liveness: each waiting process will eventually enter its critical section

does not hold on all paths, but almost surely

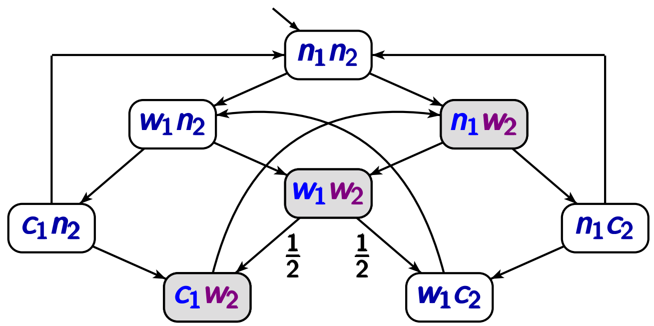
Properties of the randomized MUTEX



Suppose process **2** is waiting.

What is the probability that process **2** enters its critical section within the next **3** steps **?**

Properties of the randomized MUTEX



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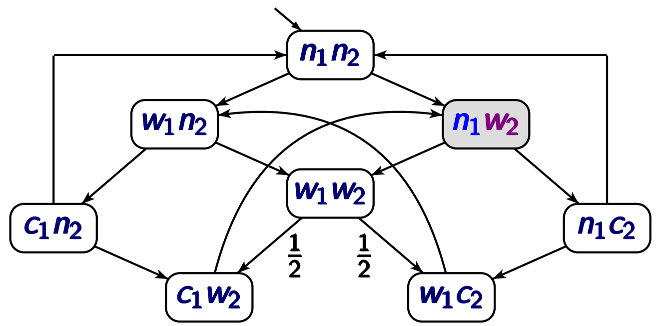
... depends ...

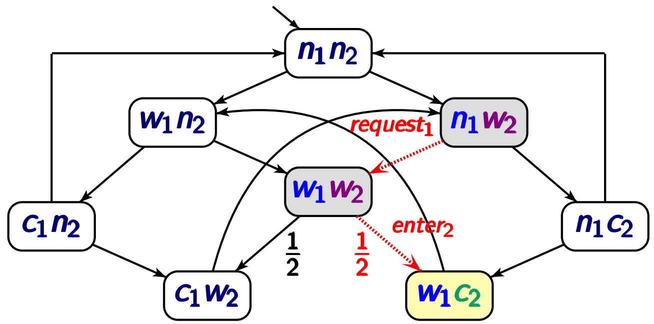
Randomized mutual exclusion protocol



Suppose the current state is $\langle n_1, w_2 \rangle$.

Randomized mutual exclusion protocol

PMC-74

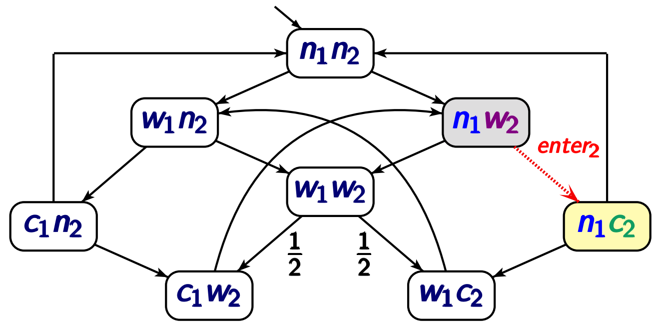


The probability that process **2** enters its critical section within the next **3** steps is:

$\frac{1}{2}$ if process **1** is scheduled in state $\langle n_1, w_2 \rangle$

Randomized mutual exclusion protocol

PMC-74



The probability that process **2** enters its critical section within the next **3** steps is:

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1 if process **2** is scheduled in state $\langle n_1, w_2 \rangle$

Reasoning about probabilities in MDP

PMC-77

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PMC-77

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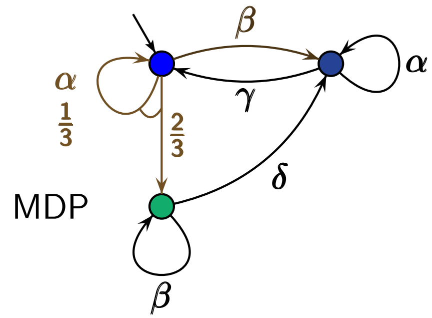

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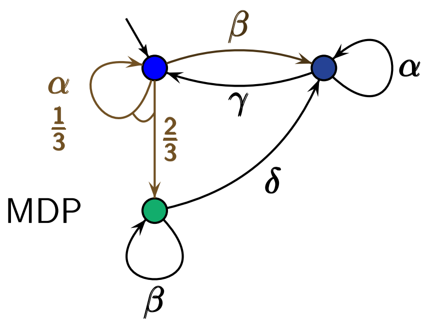

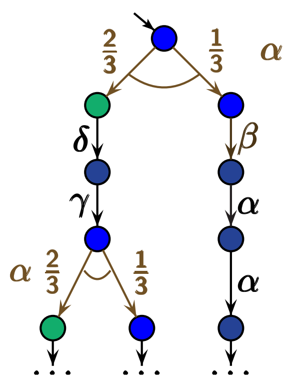

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↑

yields a notion of probability measure $\mathbf{Pr}^D$ on measurable sets of infinite paths

Discrete-time Markovian models

Part 1: discrete-time Markov chains (DTMC)

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3. fairness

PCTL* over MDPs [Bianco/de Alfaro'95]

- syntax of state and path formulas as for PCTL* over Markov chains
- probability operator $\mathbb{P}_I(\ldots)$ ranges over all schedulers

PCTL* over MDPs [Bianco/de Alfaro'95]

state formulas:

$$\Phi \ ::= \ true \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \mathbb{P}_I(\varphi)$$

path formulas:

$$\varphi \ ::= \ \Phi \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \bigcirc\varphi \mid \varphi_1 \, \mathsf{U} \, \varphi_2$$

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given an MDP $\mathcal{M}$, define by structural induction:

- a satisfaction relation $\models$ for states s in $\mathcal{M}$ and **PCTL*** state formulas
- a satisfaction relation $\models$ for infinite paths π in $\mathcal{M}$ and **PCTL*** path formulas

Satisfaction relation for PCTL* state formulas

$s \models \mathit{true}$		
$s \models a$	iff	$a \in L(s)$
$s \models \Phi_1 \wedge \Phi_2$	iff	$s \models \Phi_1$ and $s \models \Phi_2$
$s \models \neg\Phi$	iff	$s \not\models \Phi$
$s \models \mathbb{P}_I(\varphi)$	iff	for all schedulers D:
		$Pr^D\{\pi \in Paths(s) : \pi \models \varphi\} \in I$

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probability measure in the Markov chain induced by D

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probability measure in the Markov chain induced by D

semantics of path formulas as for Markov chains

PCTL* model checking for MDP

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given: MDP $\mathcal{M} = (S, Act, P, AP, L, s_0)$
PCTL* state formula Φ

task: check whether $s_0 \models \Phi$

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main procedure as for PCTL* over Markov chains:

recursively compute the satisfaction sets

$$Sat(\Psi) \;=\; \{\, s \in S : s \models \Psi \,\}$$

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treatment of the propositional logic fragment: $\surd$

Treatment of probability operator

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upper probability bounds $\mathbb{P}_{\leqslant p}(\varphi)$ or $\mathbb{P}_{<p}(\varphi)$

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- compute the maximal probabilities for φ

  $$\Pr^{\mathcal{M}}_{\max}(s, \varphi) \;=\; \sup_{D} \Pr^{D}\{\, \pi \in Paths(s) : \pi \models \varphi \}$$

  for all states s

Treatment of probability operator

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there exists optimal
finite-memory schedulers

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  for all states s
- return $\{\, s \in S : \Pr^{\mathcal{M}}_{\max}(s, \varphi) \leqslant p \,\}$

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  for all states s

- return $\{\, s \in S \,:\, \Pr^{\mathcal{M}}_{\max}(s,\varphi) \leqslant p \,\}$

lower probability bounds $\mathbb{P}_{\geqslant p}(\varphi)$ or $\mathbb{P}_{>p}(\varphi)$

analogous, but minimal probabilities for φ

Treatment of probability operator

upper probability bounds $\mathbb{P}_{\leqslant p}(\varphi)$ or $\mathbb{P}_{<p}(\varphi)$

compute the maximal probabilities for φ

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special case: $\varphi = \Diamond\Psi$

↑ reachability condition

Treatment of probability operator

upper probability bounds $\mathbb{P}_{\leqslant p}(\varphi)$ or $\mathbb{P}_{<p}(\varphi)$

compute the maximal probabilities for φ

$$\Pr^{\mathcal{M}}_{\max}(s,\varphi) \;=\; \max_{D} \Pr^{D}\{\,\pi \in Paths(s) \,:\, \pi \models \varphi\}$$

special case: $\varphi = \Diamond\Psi$

compute $\Pr^{\mathcal{M}}_{\max}(s,\Diamond\Psi)$ by solving a linear program

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general case:

via determininistic automaton $\mathcal{A}$ for φ and maximal reachability probabilities in $\mathcal{M} \times \mathcal{A}$

Maximal reachability probabilities

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given: MDP $\mathcal{M}$ with state space S
set $T \subseteq S$ of goal states

task: compute $x_s = \Pr^{\mathcal{M}}_{\max}(s, \Diamond T) = \max_D \Pr^D(s, \Diamond T)$

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The vector $(x_s)_{s \in S}$ is the least solution in $[0,1]$ of the equation system:

$$\begin{aligned} x_s &= 1 && \text{if } s \in T \\ x_s &= \max_{\alpha} \sum_{s' \in S} P(s, \alpha, s') \cdot x_{s'} && \text{if } s \notin T \end{aligned}$$

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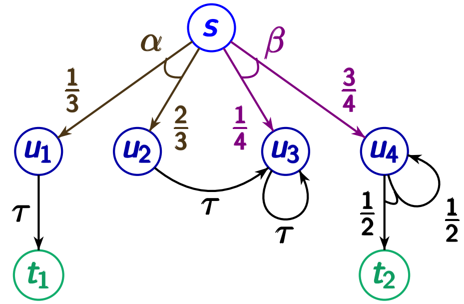

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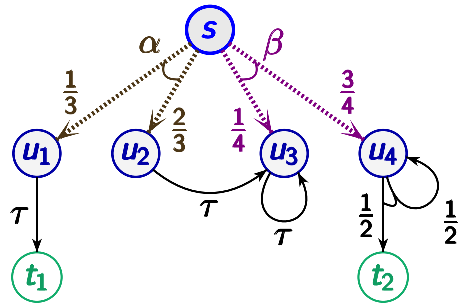

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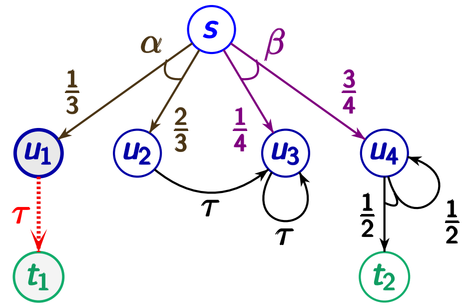
$$x_s = \max\{ \tfrac{1}{3}x_{u_1} + \tfrac{2}{3}x_{u_2}, \ \tfrac{1}{4}x_{u_3} + \tfrac{3}{4}x_{u_4} \}$$

action α action β



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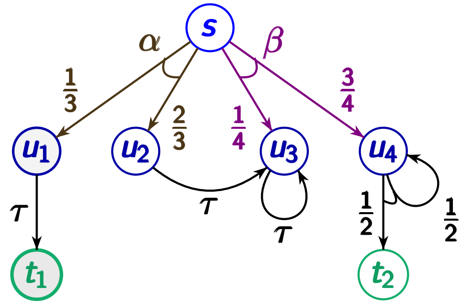
$$x_{u_1} = x_{t_1}$$

unique successor of u_1



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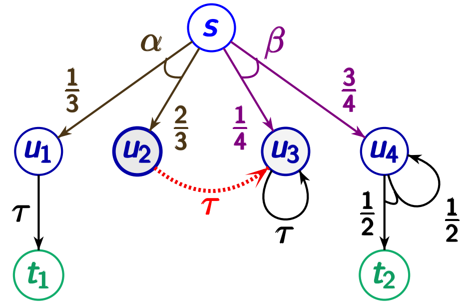
$$x_{u_1} = x_{t_1} = 1$$

↑ goal state $t_1 \in T$



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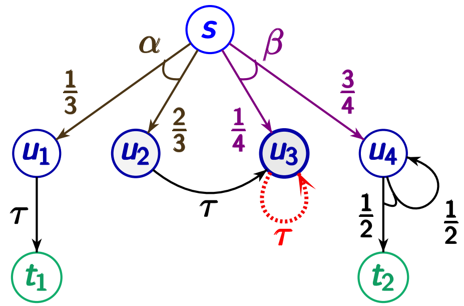
$$x_{u_1} = x_{t_1} = 1$$

$$x_{u_2} = x_{u_3}$$

↑ unique successor of state u_2



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$$x_{u_1} = x_{t_1} = 1$$

$$x_{u_2} = x_{u_3} = 0$$ least solution of $x_{u_3} = x_{u_3}$

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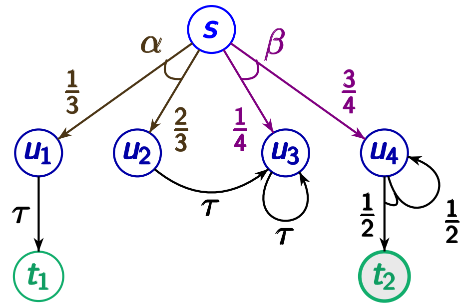

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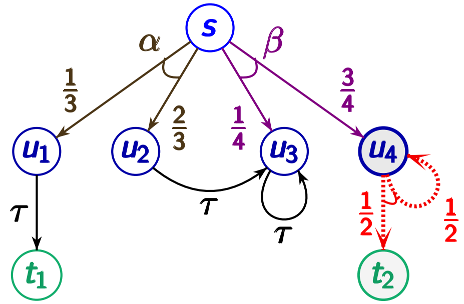

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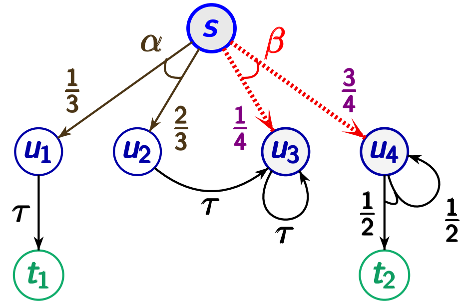

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Maximal reachability probabilities

The vector $(x_s)_{s \in S}$ where $x_s = \Pr_{\max}(s, \lozenge T)$ is the least solution in $[0,1]$ of

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Maximal reachability probabilities

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$$x_s = 1 \quad \text{if } s \in T$$
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if $s \notin T$ and T is reachable from s

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graph algorithms

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↑ can be rephrased as a linear program

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where $\sum_{s \in S^?} x_s$ is minimal

Maximal reachability probabilities

PMC-87

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linear program

Maximal probabilities for limit properties

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PMC-89

given: MDP $\mathcal{M} = (S, P, \ldots)$
prefix-independent limit property E for paths

task: compute $\Pr_{\max}(s, E)$

↑

$$\max_D \Pr^D \{ \pi \in Paths(s) : \pi \in E \}$$

Maximal probabilities for limit properties

given: MDP $\mathcal{M} = (S, P, \ldots)$
prefix-independent limit property E for paths

task: compute $\mathbf{Pr}_{\mathbf{max}}(s, E)$

i.e., there exists subsets $T_1, \ldots, T_k$ of S s.t.
for all paths π in $\mathcal{M}$:

$$\pi \models E \quad \text{iff} \quad \exists i \in \{1, \ldots, k\}.\ \inf(\pi) = T_i$$

where $\inf(s_0\, s_1\, s_2 \ldots) = \{t \in S : \overset{\infty}{\exists}\, i \geqslant 0.\ s_i = t\}$

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... relies on an analysis of end components

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(3) ...

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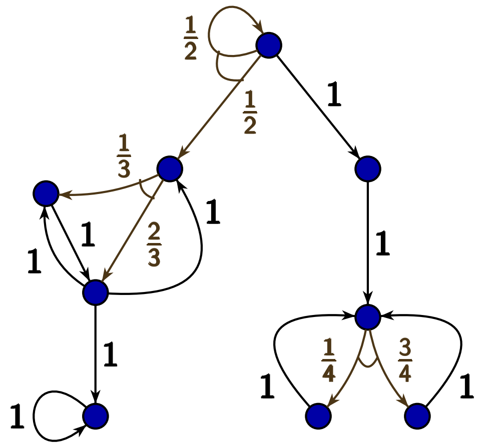
(3) the underlying graph is strongly connected

Example: end components



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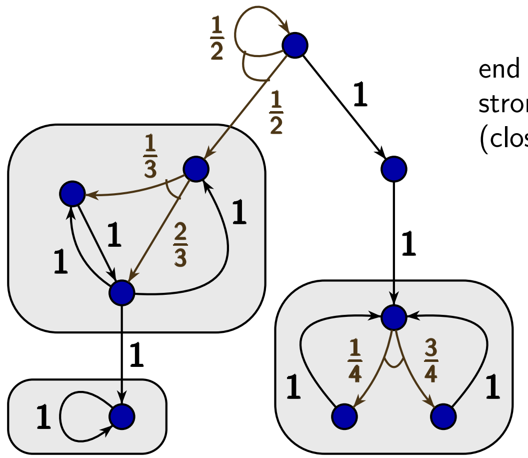

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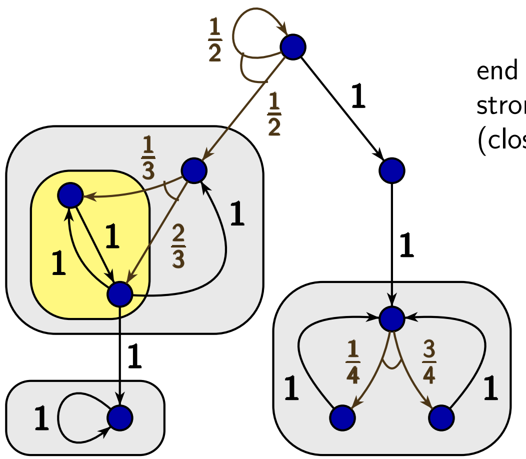

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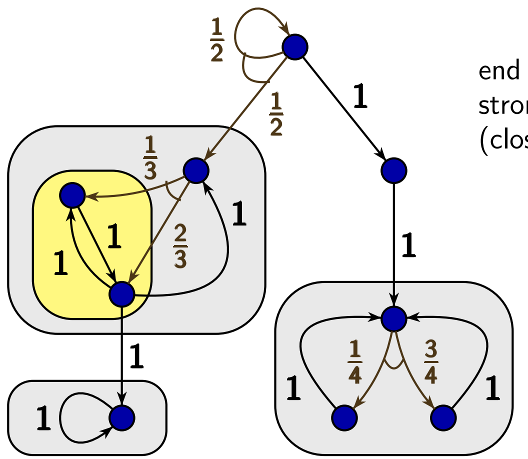
Long-run behavior of MDPs

For all schedulers ***D***, almost surely an end component will be reached and all its states visited infinitely often.



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Then: $Pr_{max}(s, E) = Pr_{max}(s, \Diamond A)$ where

$$A = \bigcup \{ T_i : T_i \text{ constitutes an end component} \}$$

PCTL* model checking for MDP

given: MDP $\mathcal{M} = (S, Act, P, \ldots)$
PCTL* state formula $\mathbb{P}_{\leqslant p}(\varphi)$

task: compute $Sat(\mathbb{P}_{\leqslant p}(\varphi))$

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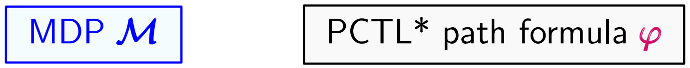
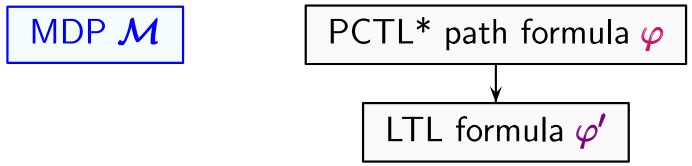
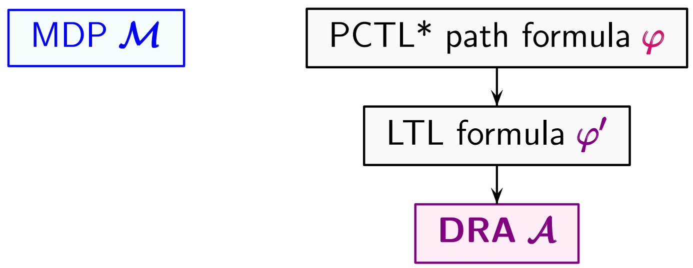
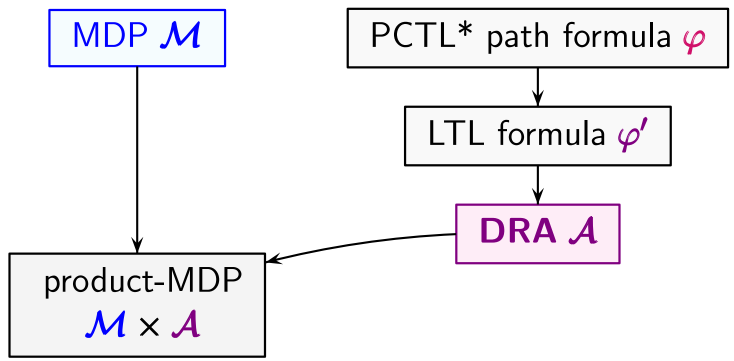
using DRA $\mathcal{A}$ for φ and linear program for $\mathcal{M} \times \mathcal{A}$

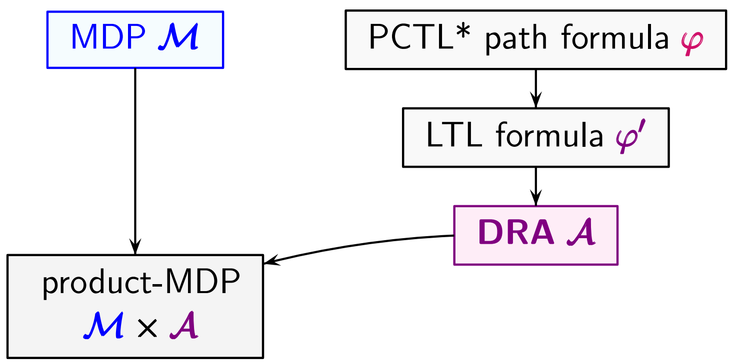
DRA: deterministic Rabin automaton











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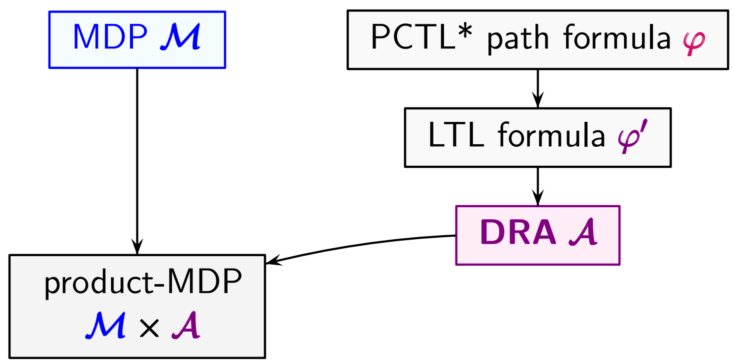

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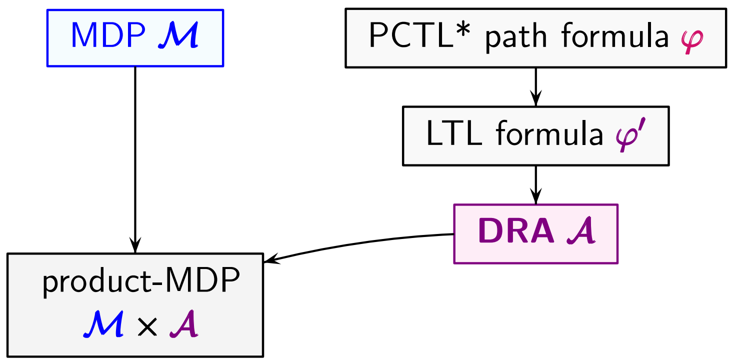
initial state in the product, if $\mathcal{M}$ starts in s,
i.e., $init_s = \delta(q_0, L(s))$



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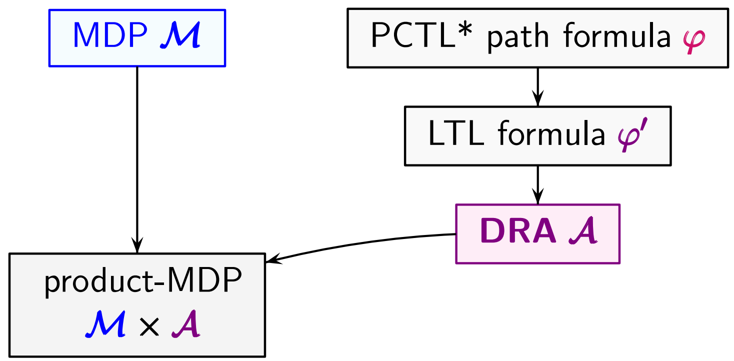
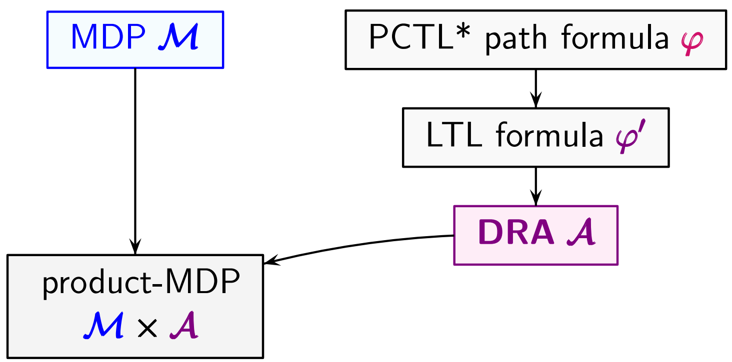
acceptance condition of $\mathcal{A}$



$$\begin{aligned}\Pr^{\mathcal{M}}_{\max}(s,\varphi) &= \Pr^{\mathcal{M}\times\mathcal{A}}_{\max}\big(\langle s, init_s\rangle, \bigvee_i (\Diamond\Box\neg L_i \wedge \Box\Diamond U_i)\big)\\ &= \Pr^{\mathcal{M}\times\mathcal{A}}_{\max}\big(\langle s, init_s\rangle, \Diamond accEC\big)\end{aligned}$$

union of all accepting end components



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Lower probability bounds

given: MDP $\mathcal{M} = (S, Act, P, \ldots)$

PCTL* star formula $\mathbb{P}_{\geqslant p}(\varphi)$

task: compute $Sat\big(\mathbb{P}_{\geqslant p}(\varphi)\big)$

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For each state s and PCTL* path formula φ:

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Complexity of PCTL/PCTL* model checking

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	PCTL	PCTL*
Markov chain		
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tools: PRISM (Oxford), MRMC (Aachen), ...

Discrete-time Markovian models

Part 1: discrete-time Markov chains (DTMC)

1. basic definitions
2. probabilistic computation tree logic (PCTL/PCTL*)
3. expected rewards

Part 2: Markov decision processes (MDP)

1. basic definitions
2. PCTL/PCTL* model checking
3. fairness ⟵

Why MDP with fairness ?

Markov decision processes

- extend Markov chains by nondeterminism
- modelling asynchronous distributed systems by interleaving

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verification of liveness properties
(qualitative or quantitative)
often requires fairness assumptions

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> verification of liveness properties
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e.g., strong process fairness:

$\Box\Diamond$ process P is enabled $\longrightarrow$ $\Box\Diamond$ actions of process P are taken

Markov decision processes

PMC-110

- extend Markov chains by nondeterminism
- modelling asynchronous distributed systems by interleaving

↑

> verification of liveness properties
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> often requires fairness assumptions

general case: fairness assumptions impose restrictions on the resolution of nondeterminism to rule out unrealistic behaviors

MDP with fairness

given: MDP $\mathcal{M} = (S, Act, P, \ldots)$

A fairness assumption $\mathcal{F}$ for $\mathcal{M}$ is a conjunction of limit properties of the form:

$\Box\Diamond Y$	unconditional fairness
$\Box\Diamond X \rightarrow \Box\Diamond Y$	strong fairness
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where $X, Y \subseteq S$

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PMC-111

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here: just state-based fairness conditions

action-based fairness conditions can be encoded, e.g.,

$\Box\Diamond enabled(A) \rightarrow \Box\Diamond taken(A)$ where $A \subseteq Act$

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Scheduler D for $\mathcal{M}$ is called $\mathcal{F}$-fair iff

$\Pr^D(s, \mathcal{F}) = 1$ for all (reachable) states s

Realizability of fairness assumptions

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↑ union of all maximal end components that contain a sub-component T where $\mathcal{F}$ holds,
i.e., $X \cap T = \varnothing$ or $Y \cap T \neq \varnothing$

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poly-time algorithm for computing *FairMEC*:

... recursive computation of maximal end components in sub-MDPs

PCTL* with fairness

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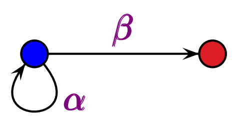

$\mathcal{F} = \Box\Diamond blue \rightarrow \Box\Diamond red$

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↑ union of all fair end components without any b-state

PCTL* model checking for MDP with fairness

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given: MDP $\mathcal{M} = (S, Act, P, AP, L, s_0)$
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PCTL* state formula Φ

task: check whether $s_0 \models_{\mathcal{F}} \Phi$

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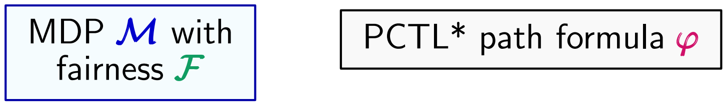
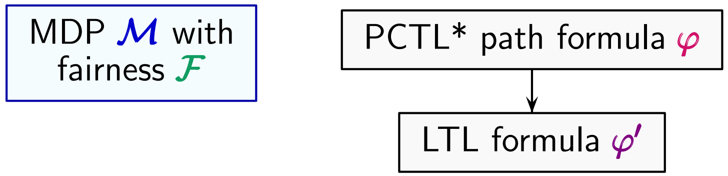
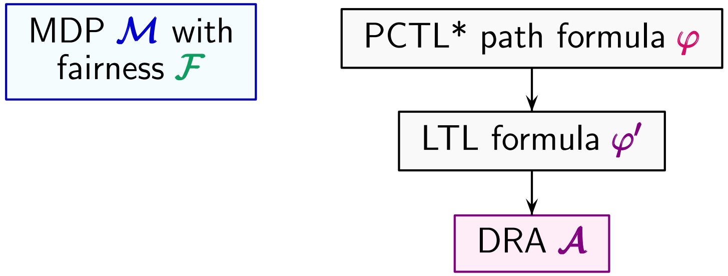
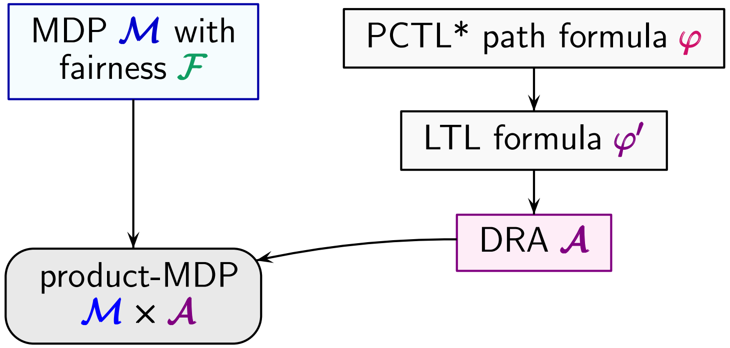
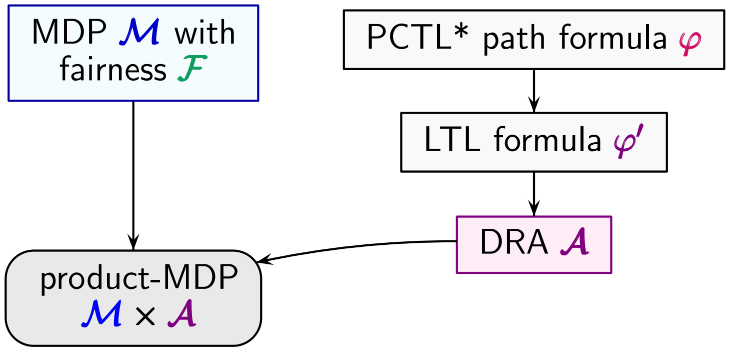
treatment of the propositional logic fragment: √











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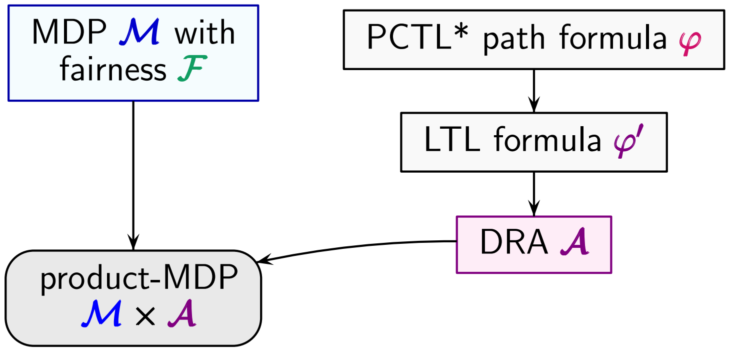

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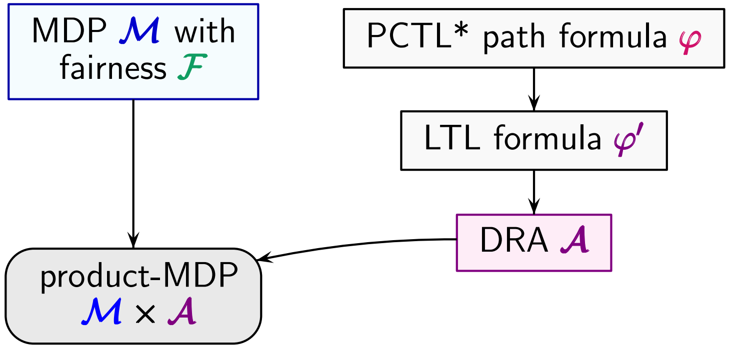
D ranges over all fair schedulers for $\mathcal{M}$

E ranges over all fair schedulers for $\mathcal{M} \times \mathcal{A}$



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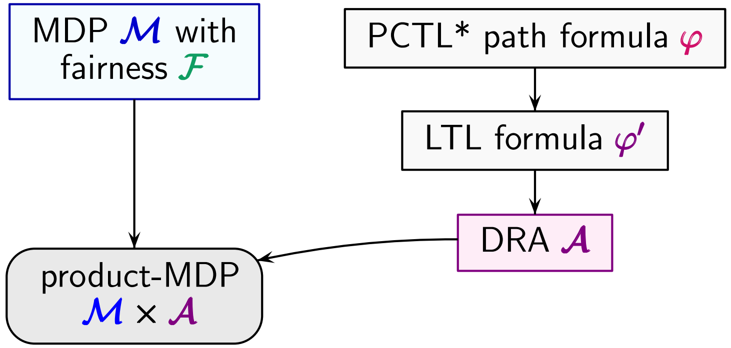

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Conclusion

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- model checking for systems with discrete probabilities
  - ∗ techniques for verifying non-probabilistic systems
    graph algorithms, automata, ...
  - ∗ numerical methods for solving
    linear equation systems (Markov chains)
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  - numerical methods for solving
    linear equation systems (Markov chains)
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- tool support
  - symbolic MTBDD-based PRISM [Kwiatk. et al]
  - abstraction, bisimulation MRMC [Katoen et al]
  - abstraction-refinement PASS [Hermanns et al]
    RAPTURE [d'Argenio et al]
  - partial order reduction LiQuor [Ciesinski et al]

    ⋮

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PMC-CONC

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- very active research field
  - continuous-time and -space
  - probabilistic real-time/hybrid systems
  - stochastic games
  - various techniques for state explosion problem
  - applications in system biology, security, ...

    ⋮